Homomorphic Signatures for Polynomial Functions

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Homomorphic Encryption

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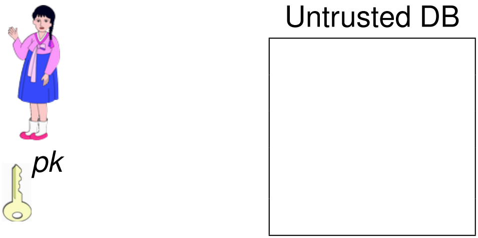

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$\xrightarrow[\text{grades}]{\text{encrypted}}$

Untrusted DB

Student	CT
Adam	c_1
Becky	c_2
$\vdots$	$\vdots$
Kevin	c_k

$c_i =$ encryption of ith score

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- Validity: c decrypts to the correct mean.
- Security: no adversary can obtain any info about scores.
- Length efficiency: c is short.
- Privacy: decrypted mean reveals nothing else about data.

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σ_1 = signature on
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4. Privacy: σ reveals nothing about data other than the mean.

More generally: $\mathcal{F}$-homomorphic signatures

- $\mathcal{F}$ is a set of "admissible" functions on messages.
- τ is a "tag" tying together data from the same set. (like a filename)
    - prevents mixing of data from different sets
- Given *pk*, admissible function $f \in \mathcal{F}$, and signatures on data

$$m_1, \ldots, m_k,$$

anyone can compute a valid signature on

$$(\tau,\ f(m_1, \ldots, m_k),\ \omega(f)),$$

where $\omega(f)$ is an "encoding" or "digest" of the function f.

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Subsets	Message redaction

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Specifically, we construct secure, length-efficient, $\mathcal{F}$-homomorphic signatures for

$$\mathcal{F} = \{\text{polynomials of bounded degree with small coefficients}\}$$

Application: Least Squares Fits

Least squares fits — the basics

For a data set $\{(x_i, y_i)\}_{i=1}^{k}$, the degree d least squares fit is a polynomial

$$f(x) = c_0 + c_1 x + \cdots + c_d x^d$$

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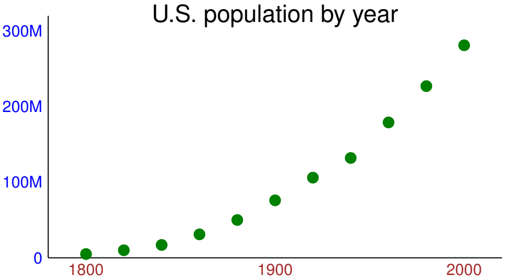

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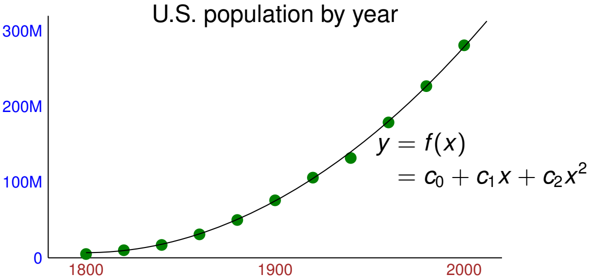

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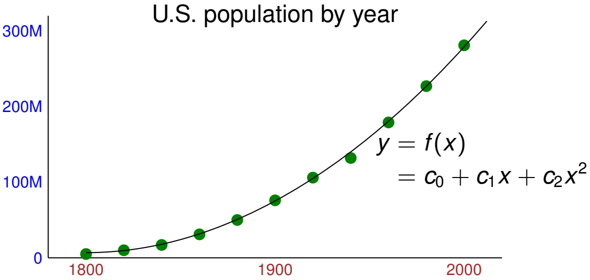
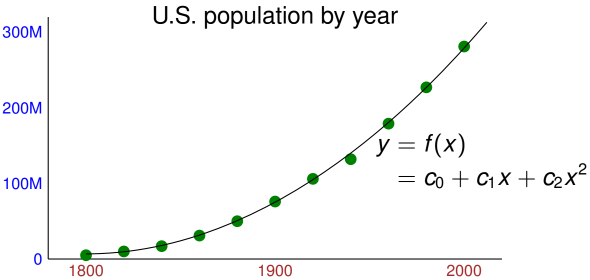


Formula:

$$\vec{c} = (X^t X)^{-1} X^t \vec{y}$$

$\vec{c}$ = vector of coefficients of $f(x)$,
X = Vandermonde matrix of x values,
$\vec{y}$ = vector of y values.

Authenticating a least-squares fit (x-values only)



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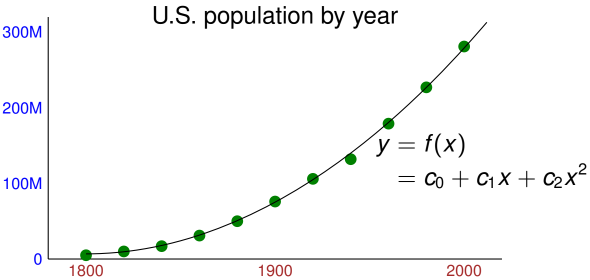

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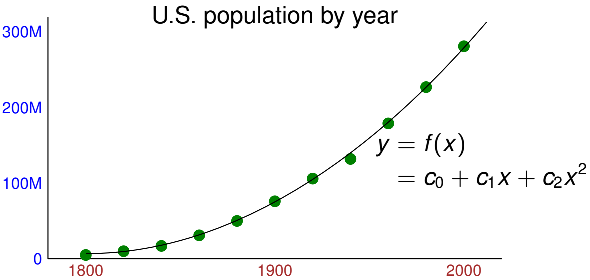

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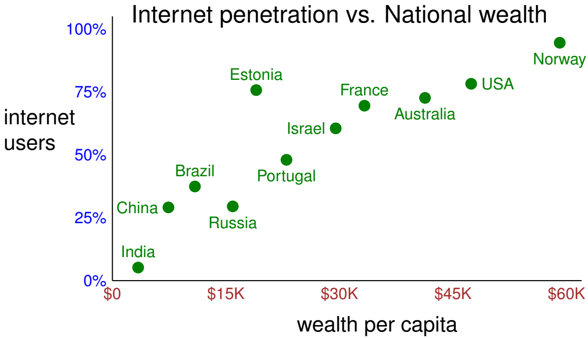
- Census bureau stores signed population counts on server using linearly homomorphic signature.
- Server can authenticate coefficients of least-squares fit.

Authenticating a least-squares fit (x and y values)



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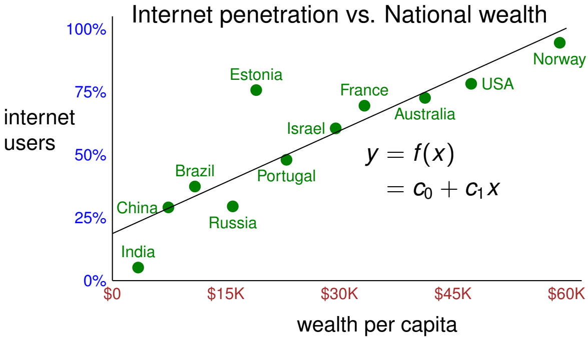

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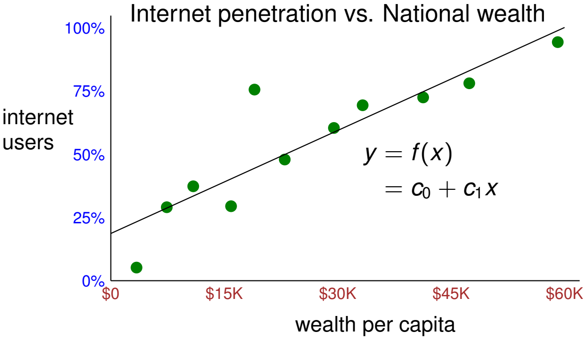

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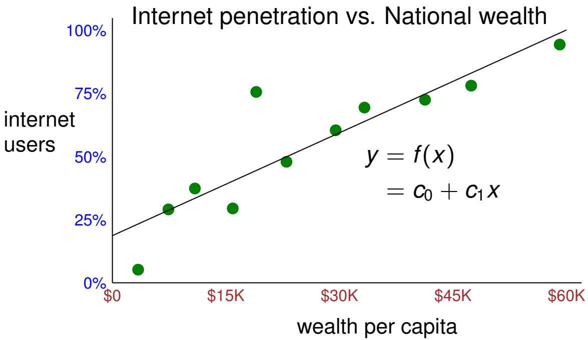

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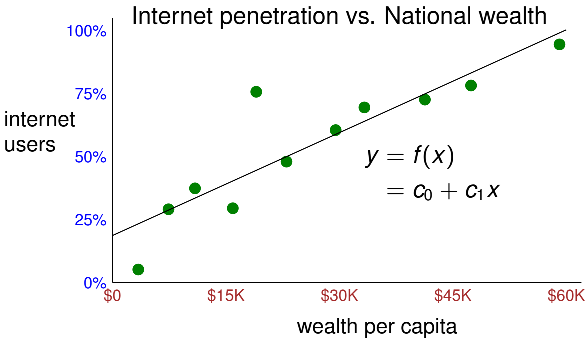
- However: $\det(X^t X) \cdot c_j$ are polynomial functions of x and y.



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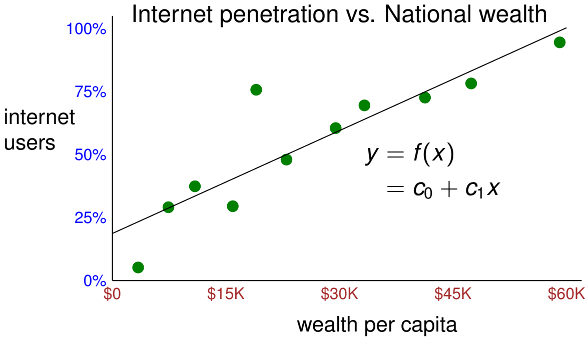

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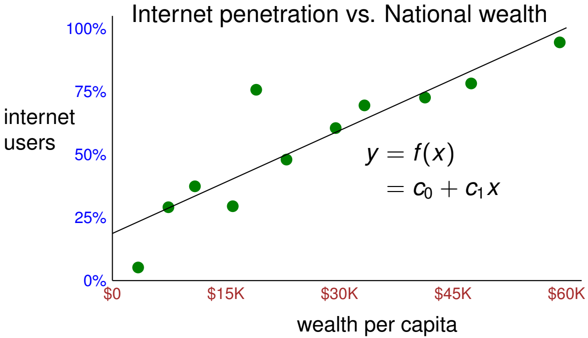

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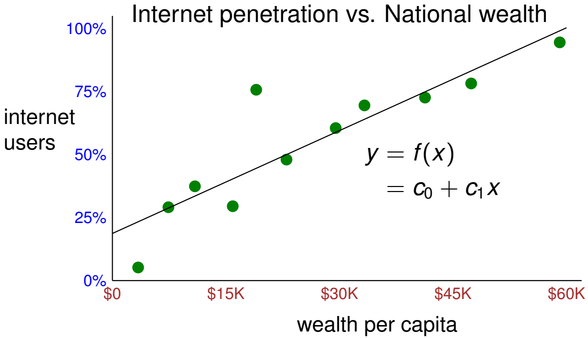

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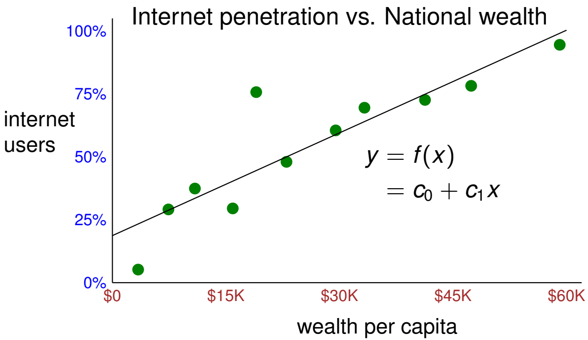

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- Linear fit can be computed using degree 3 polynomials.

Linearly Homomorphic Signatures

Building block: GPV Signatures

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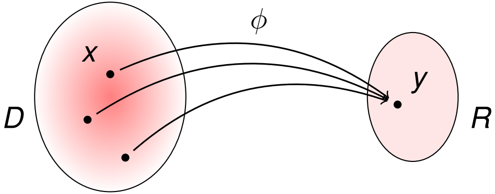

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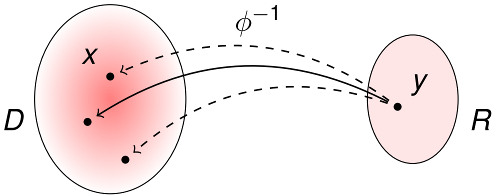

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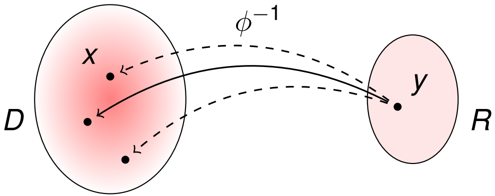
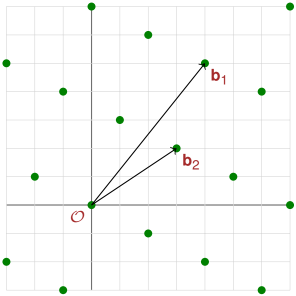


- "Hash and sign:" $pk = \phi$, $sk = \phi^{-1}$, hash $H\colon \{0,1\}^* \to R$

$$\begin{aligned}\mathrm{Sign}(m) &:= \phi^{-1}(H(m))\\ \mathrm{Verify}(\sigma) &: \phi(\sigma) \stackrel{?}{=} H(m)\end{aligned}$$

GPV Signatures, concretely

- $\Lambda \subset \mathbb{Z}^n$ a lattice (full-rank additive subgroup), defined by basis.



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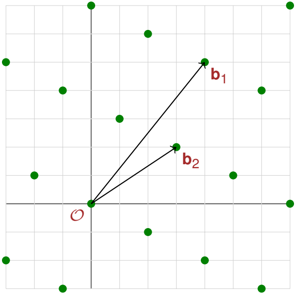

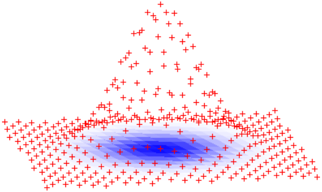

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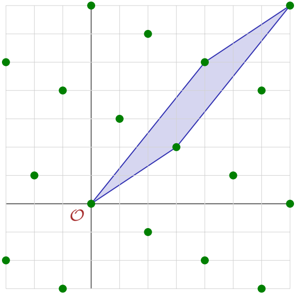

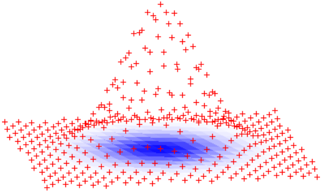

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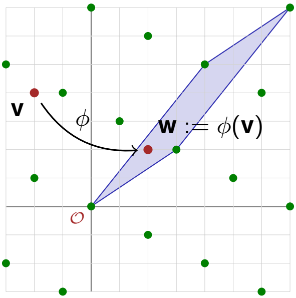

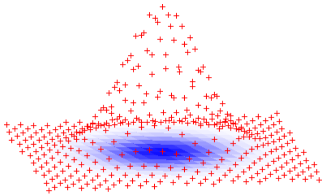

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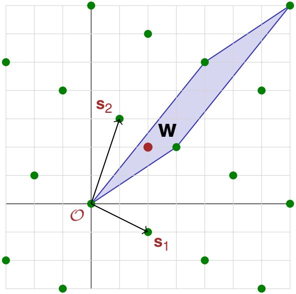

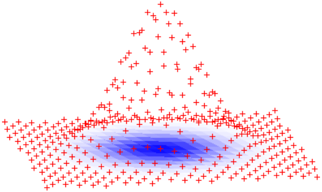

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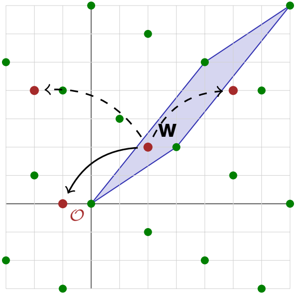
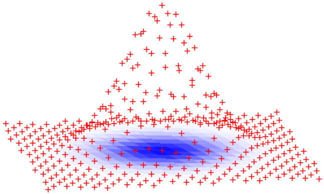
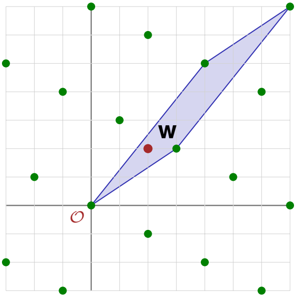
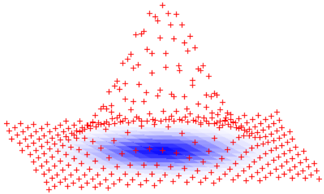




GPV Signatures, concretely

- $\Lambda \subset \mathbb{Z}^n$ a lattice (full-rank additive subgroup), defined by basis.
- $D =$ short vectors in $\mathbb{Z}^n$, with Gaussian distribution.
- $R = \mathbb{Z}^n / \Lambda$ (fix unique representatives)
- Trapdoor function $\phi \colon \mathbf{v} \mapsto (\mathbf{v} \bmod \Lambda)$ i.e., move $\mathbf{v}$ into a fundamental parallelepiped.
- GPV: algorithm to sample short vectors in $\phi^{-1}(\mathbf{w}) = \Lambda + \mathbf{w}$ given a "short" basis of Λ.
- Sampling from $\Lambda + \mathbf{w}$ without short basis is hard. (How hard depends on Gaussian parameter.)





Linearly Homomorphic Signatures: Key Idea #1

Idea: instead of hashing the messages to $R = \mathbb{Z}^n/\Lambda$,
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- For $a, b \in \mathbb{Z}$, define signature on $am_1 + bm_2$ to be
  $$\sigma := a\sigma_1 + b\sigma_2.$$
  $\Rightarrow \sigma$ is short (if a, b small), $\sigma \bmod \Lambda = am_1 + bm_2$.

Problem: Removing hash function destroys security

Valid signature doesn't imply function was computed correctly.

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sk

signed
grades →

Untrusted DB

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⋮	⋮	⋮
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← mean?

pk

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- σ authenticates 18, but 18 is not the mean!

Linearly Homomorphic Signatures: Key Idea #2

Use a second lattice to authenticate functions:

- $\Lambda_2 \subset \mathbb{Z}^n$ distinct from $\Lambda_1 := \Lambda$.
  - require $\Lambda_1 + \Lambda_2 = \mathbb{Z}^n$
- Map $\phi_2 \colon \mathbb{Z}^n \to \mathbb{Z}^n / \Lambda_2$ given by $\phi_2(\mathbf{v}) := \mathbf{v} \bmod \Lambda_2$.

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If "encoding" $\omega(\cdot)$ is linear, (i.e., $\omega(f) + \omega(g) = \omega(f+g)$)
then signature is a linear operator on the space of functions.

How do we encode functions?

Ingredients:

- k := number of messages input to a function.
- τ := "tag" that ties together messages in same data set.
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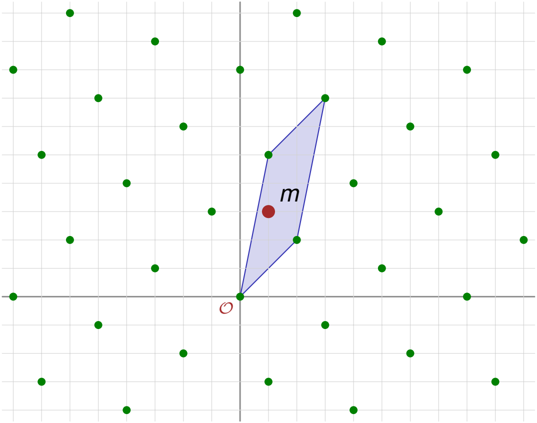
- c_i are small integers.
- "encoding" $\omega(f)$ much shorter than description of f.

“Intersection method” binds messages to functions

- Messages $m \in \mathbb{Z}^n/\Lambda_1$.



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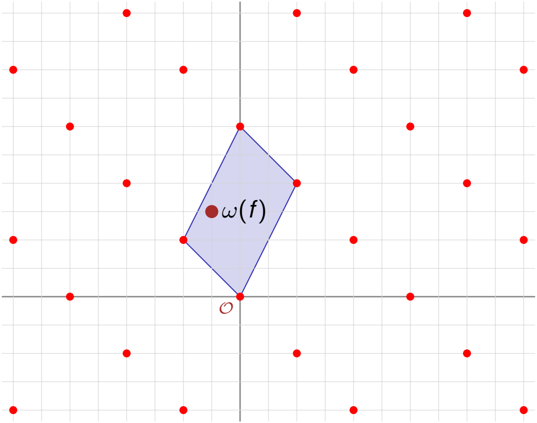

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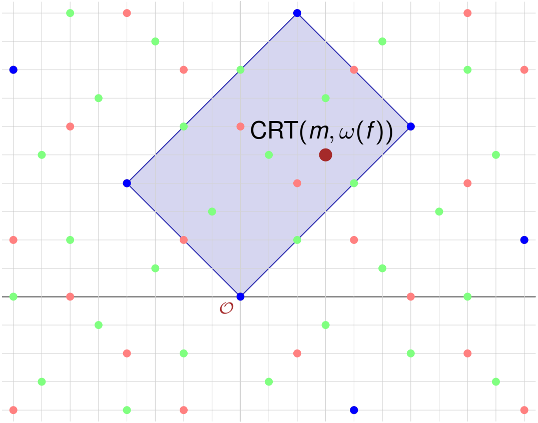

$$\mathrm{CRT}(m, \omega(f))$$
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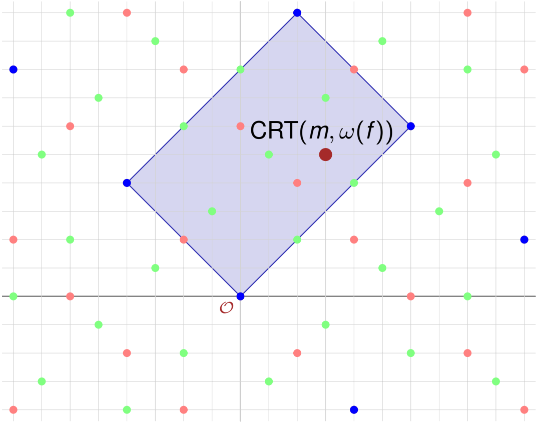

$$\begin{aligned} &\mathrm{CRT}(m, \omega(f)) \\ &= m \bmod \Lambda_1 \\ &= \omega(f) \bmod \Lambda_2 \end{aligned}$$

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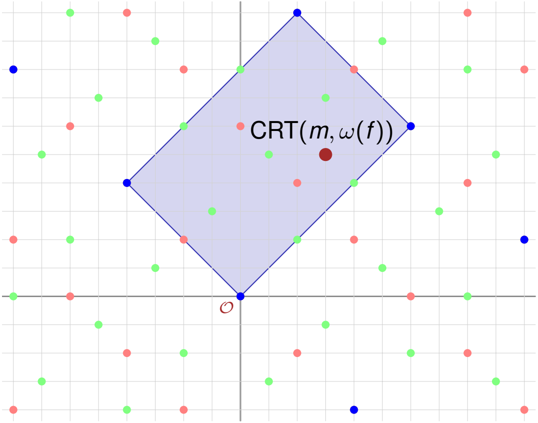

$$\mathrm{CRT}(m, \omega(f)) = m \bmod \Lambda_1 = \omega(f) \bmod \Lambda_2$$

$$\begin{aligned} \mathrm{Sign}(m) &:= \text{short vector in } (\Lambda_1 \cap \Lambda_2) + \mathrm{CRT}(m, \omega(f)) \\ \mathrm{Verify}(\sigma) &:= 1 \quad \text{iff} \quad (\sigma \bmod \Lambda_1 = m) \text{ and } (\sigma \bmod \Lambda_2 = \omega(f)) \\ &\qquad \text{and } \sigma \text{ is short} \end{aligned}$$

Linearly homomorphic signature scheme

- KeyGen(n):
    - pk = Lattices $\Lambda_1, \Lambda_2 \subset \mathbb{Z}^n$, Gaussian parameter β
    - sk = short basis of $\Lambda_1 \cap \Lambda_2$
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  1. $\sigma \bmod \Lambda_1 = m$,
  2. $\sigma \bmod \Lambda_2 = \omega(f) = \sum c_i\alpha_i$,
  3. σ sufficiently short.

Concrete example

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Same functionality as *network coding* signatures [BFKW09,GKKR10], except p can be small (even $p = 2$).

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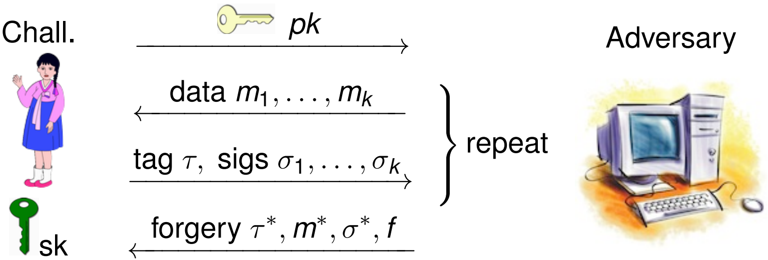

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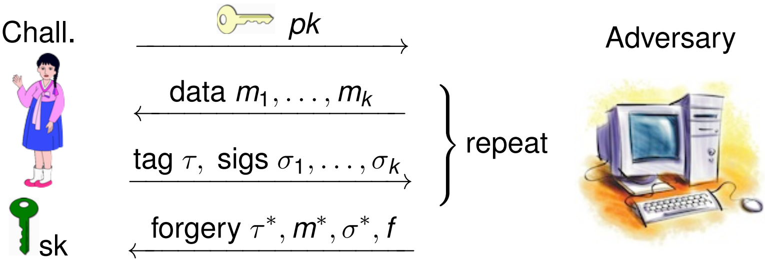

Adversary wins if f admissible, σ^* verifies for (τ^*, m^*, f), and

1. τ^* not obtained in response to a query, or
2. $\tau^* = \tau$ for query $(m_1, \ldots, m_k)$, and $m^* \neq f(m_1, \ldots, m_k)$.

Security Theorem

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Proof outline

1. Given a "challenge" Λ_2, answer signature queries without a basis of $\Lambda_1 \cap \Lambda_2$.
2. Use adversary's forgery to produce a short vector in Λ_2.

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  1. Use basis of Λ_1 to compute short vectors $\sigma_i \in \Lambda_1 + m_i$;
  2. Set $\alpha_i := \sigma_i \bmod \Lambda_2 \in \mathbb{Z}^n/\Lambda_2$.
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- For certain parameter choices, α_i are statistically close to uniform in $\mathbb{Z}^n/\Lambda_2$.
- Simulation is indistinguishable from real system.

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Conclusion: $\sigma^* - \sigma$ is (1) nonzero, (2) in Λ_2, (3) short.

- If τ^* not obtained from a query, sign random messages m_i and perform same analysis.

Privacy

Privacy property: derived signature on $f(m_1, \ldots, m_k)$ reveals nothing about $m_1, \ldots, m_k$ beyond value of f.

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Specifically: given data sets

$$\vec{m} = (m_1, \ldots, m_k), \qquad \vec{m}' = (m'_1, \ldots, m'_k)$$

and admissible function f with

$$f(\vec{m}) = f(\vec{m}'),$$

even unbounded adversary cannot distinguish derived signature on $f(\vec{m})$ from derived signature on $f(\vec{m}')$.

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Proof idea

- Distribution of derived signature on $f(\vec{m})$ depends only on f and $f(\vec{m})$, not on $\vec{m}$.
- If $f(\vec{m}) = f(\vec{m}')$, distributions of derived sigs are identical.

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Key technical fact [BF11]: distribution of linear combination of discrete Gaussian samples is also discrete Gaussian.

- Sigs on m_i sampled from discrete Gaussian distribution, derived sigs are linear combinations.

Polynomially Homomorphic Signatures from Ideal Lattices

Extending the system

Linearly homomorphic scheme: messages in $\mathbb{Z}^n/\Lambda_1$, functions "encoded" in $\mathbb{Z}^n/\Lambda_2$, signatures are short vectors in $\mathbb{Z}^n$.

$\phi_i \colon \mathbb{Z}^n \to \mathbb{Z}^n/\Lambda_i$ given by $\mathbf{v} \mapsto (\mathbf{v} \bmod \Lambda_i)$ is a linear map, so we can add either before or after applying ϕ_i.

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- Can authenticate polynomial functions on messages.

Setup for polynomial system [G09]

Fix monic, irreducible $F(x) \in \mathbb{Z}[x]$ of degree n.

- $R := \mathbb{Z}[x]/(F(x))$ gives a ring structure on $\mathbb{Z}^n$:

$$\begin{array}{ccc} \text{(coordinates of vectors)} & \leftrightarrow & \text{(coefficients of polynomials mod } F\text{)} \\ (a_0, \ldots, a_{n-1}) & \leftrightarrow & a_0 + a_1 x + \cdots + a_{n-1} x^{n-1} \end{array}$$

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- Message space is $R/\mathfrak{p} = \mathbb{F}_p$.
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$\Lambda_2 =$ prime ideal $\mathfrak{q}$; polynomials “encoded” in $R/\mathfrak{q} = \mathbb{F}_q$:

- Hash function $H\colon \{0,1\}^* \to \mathbb{F}_q^k$ maps $\tau \mapsto (\alpha_1, \ldots, \alpha_k)$.
- “Encode” f by $\omega(f) := f(\alpha_1, \ldots, \alpha_k)$.
  (think of coefficients of f as small integers).

The polynomial system, concretely

- KeyGen(n):
  - $F(x) \in \mathbb{Z}[x]$ degree n
    $\Rightarrow$ ring structure on $\mathbb{Z}^n \cong R := \mathbb{Z}[x]/(F(x))$.
  - pk = prime ideals $\mathfrak{p}, \mathfrak{q} \subset R$, Gaussian parameter β
  - sk = short basis of $\mathfrak{p} \cap \mathfrak{q} = \mathfrak{p} \cdot \mathfrak{q}$
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[G09,G10]: parameter γ_F measures how much multiplication in R increases length:

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- Product of d elements of length $< \beta$ has length $< \gamma_F^{d-1} \beta^d$.
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Lots of $F(x)$ have small γ_F:

- e.g., cyclotomic polynomials $\Phi_\ell(x)$, ℓ prime or $\ell = 2^a 3^b$.

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- $uv \cdot R = \mathfrak{p} \cdot \mathfrak{q}$, and

$$\mathbf{B} := \{uv,\ uv \cdot x,\ uv \cdot x^2, \ldots,\ uv \cdot x^{n-1}\}.$$

spans $\mathfrak{p} \cdot \mathfrak{q}$ and consists of short elements:

$$\|uv \cdot x^i\| \leq \|u\| \cdot \|v\| \cdot \gamma_F^2.$$

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- Signatures on original messages m_i have length $< \beta$
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- If $\beta, \gamma_F, k, y \in \text{poly}(n)$ and $d = O(1)$, then derived signature length is $\text{poly}(n)$. (p is exponential in n)
- For fixed n, bit length of derived signatures is linear in d, logarithmic in k.

What is security based on?

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Analysis of linearly homomorphic scheme also applies here:

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  - Producing a short generator of arbitrary principal $\mathfrak{q}$ is a classical problem in algorithmic number theory.
- Distribution of Smart-Vercauteren $\mathfrak{q}$ not well understood.
  - Want $\mathfrak{q}$ in distribution that admits a worst-case reduction.

Open questions

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Thank you!

Thanks also to Chris Peikert for help with graphics.