Constructing Pairing-Friendly Elliptic Curves for Cryptography

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Outline

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What is a pairing?

- Many public-key cryptographic protocols are based on the *discrete logarithm problem* (DLP) in a finite cyclic group $\mathbb{G}$:
  - Given x, y in $\mathbb{G}$, find integer a such that $y = x^a$.
  - For systems involving $\mathbb{G}$ to be secure, the DLP must be computationally infeasible.
- A *cryptographic pairing* is map

$$e : \mathbb{G} \times \mathbb{G} \to \mathbb{G}_T$$

that is

1. Bilinear: $e(x^a, y^b) = e(x, y)^{ab}$ in $\mathbb{G}_T$.
2. Nondegenerate: for any $x \in \mathbb{G}$, $x \neq 1$, there is a $y \in \mathbb{G}$ such that $e(x, y) \neq 1$ in $\mathbb{G}_T$.

How to Use a Pairing

- A cryptographic pairing maps the discrete logarithm problem in $\mathbb{G}$ to the DLP in $\mathbb{G}_T$:
  - Given x and $y = x^a$ in $\mathbb{G}$:
    1. Choose a $z \in \mathbb{G}$ with $e(x, z) \neq 1$.
    2. Compute $x' = e(x, z)$, $y' = e(y, z)$.
    3. Try to compute a from x' and $y' = x'^a$.
- The pairing solves the *Decision Diffie-Hellman Problem* in $\mathbb{G}$:
  - Given x, x^a, x^b, x^c, determine if $c = ab$.
    1. Compute $e(x, x^c) = e(x, x)^c$ and $e(x^a, x^b) = e(x, x)^{ab}$.
    2. $ab = c$ if and only if the two pairings are equal in $\mathbb{G}_T$.

Applications of Pairings

- Attack on discrete logarithm problem for supersingular elliptic curves (Menezes-Okamoto-Vanstone).
  - Map discrete log on elliptic curve to easier discrete log in finite field.
- One-round 3-way key exchange (Joux).
- Identity-based encryption (Sakai-Ohgishi-Kasahara; Boneh-Franklin).
- Short digital signatures (Boneh-Lynn-Shacham).
- Many other applications:
  - Group signatures, batch signatures, threshold cryptography, broadcast encryption, private information retrieval, electronic voting, etc.

Requirements for Pairings

- To be useful in applications using the Decision Diffie-Hellman property, we need:
  1. the discrete logarithm problem in $\mathbb{G}$ to be computationally infeasible,
  2. the discrete logarithm problem in $\mathbb{G}_T$ to be computationally infeasible, and
  3. the pairing to be easy to compute.

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Elliptic Curves in Cryptography

- An *elliptic curve* E over a finite field $\mathbb{F}_p$ is defined by an equation
$$E : y^2 = x^3 + ax + b \pmod{p}$$
- The set of points (x, y) on E, plus a "point at infinity" O, forms a group (usually written additively).
  - Adding two points P, Q gives a third point $R = P + Q$.
  - Adding a point to itself repeatedly gives multiplication: $P + \cdots + P$ (m times) $= mP$.
- If P has prime order r, then computing a from P and aP takes time $\sim \sqrt{r}$.
  - If r is a large prime, then discrete log in elliptic curve subgroup of order r is infeasible.

The Weil and Tate Pairings

- Let E be an elliptic curve defined over a finite field $\mathbb{F}$.
- For any integer r the *Weil pairing* $e_{r,weil}$ is a bilinear map sending pairs of points of order r to r-th roots of unity in $\overline{\mathbb{F}}$:

$$e_{r,weil} : E[r] \times E[r] \longrightarrow \mu_r.$$

- The *Tate pairing* $e_{r,tate}$ is similar:

$$e_{r,tate} : E(\mathbb{F})[r] \times E(\mathbb{F})/rE(\mathbb{F}) \longrightarrow \mathbb{F}^{\times}/(\mathbb{F}^{\times})^{r}.$$

- If r is prime and $\mathbb{F}_{p^k}$ is the smallest field containing μ_r, then both pairings take values in $\mathbb{F}_{p^k}^{\times}$.
- Tate pairing generally can be computed more efficiently.

Embedding degrees

- Elliptic curve pairings used in cryptography use curves $E/\mathbb{F}_p$ with a point of order r, and map into the rth roots of unity in $\mathbb{F}_{p^k}$.
- k is the *embedding degree* of E (with respect to r).
  - k is the smallest integer such that $r \mid p^k - 1$.
  - k is the order of p in $(\mathbb{Z}/r\mathbb{Z})^\times$.
  - Want k large enough so that discrete log in $\mathbb{F}_{p^k}^\times$ is computationally infeasible, but small enough so that pairing is easy to compute.
- r is a large prime dividing $\#E(\mathbb{F}_p)$
  - Define $\rho = \log p / \log r = \#\text{bits of } p / \#\text{bits of } r$.
  - If keys, signatures, ciphertexts, etc. are elements of $E[r]$, we want ρ small to save bandwidth.
  - If curve has prime order, $\rho = 1$.

Pairing-friendly elliptic curves

- Balasubramanian-Koblitz: If $E/\mathbb{F}_p$ is a "random" elliptic curve with an order-r subgroup, then $k \sim r$.
  - Pairing computation on random curves is totally infeasible: If $r \sim p \sim 2^{160}$, pairing is computed in field of size $2^{2^{160}}$.
- A *pairing-friendly curve* is an elliptic curve with a large prime-order subgroup ($\rho \leq 2$) and small embedding degree ($k \leq 50$).
- Problem: construct pairing-friendly elliptic curves for specified values of k and number of bits in r.

Our Goal

- Provide a supply of pairing-friendly curves suitable for many different performance and security requirements.
- Since discrete logarithm problem is easier in finite fields than on elliptic curves, finite field size $\mathbb{F}_{p^k}$ should be larger than subgroup size r.
- How much larger depends on level of security desired:

Security level (in bits)	Subgroup size r (in bits)	Extension field size p^k (in bits)	Embedding degree k	
			$\rho \approx 1$	$\rho \approx 2$
80	160	960 – 1280	6 – 8	3 – 4
112	224	2200 – 3600	10 – 16	5 – 8
128	256	3000 – 5000	12 – 20	6 – 10
192	384	8000 – 10000	20 – 26	10 – 13
256	512	14000 – 18000	28 – 36	14 – 18

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Supersingular Curves

- An elliptic curve over $\mathbb{F}_p$ ($p \geq 5$) is *supersingular* if $\#E(\mathbb{F}_p) = p + 1$.
  - Supersingular curves easy to construct; e.g., $y^2 = x^3 + 1$ for any $p \equiv 2 \pmod 3$.
- If $p \geq 5$, then supersingular curves over $\mathbb{F}_p$ have embedding degree 2.
- Supersingular curves over non-prime fields have embedding degree ≤ 6.
- To obtain other embedding degrees, we must use ordinary (i.e., non-supersingular) elliptic curves.

Ordinary Elliptic Curves: The General Strategy

To construct pairing-friendly ordinary elliptic curves:

1. Fix k, find primes p and r such that there exists a curve $E/\mathbb{F}_p$ with a subgroup of order r and embedding degree k.
2. Use Complex Multiplication method to construct the equation for E.

Outline

The Complex Multiplication Method

- Originally due to Atkin and Morain for primality testing application.
- Define the *trace* of E to be t such that $\#E(\mathbb{F}_p) = p + 1 - t$.
  - Hasse bound: if E ordinary then $|t| < 2\sqrt{p}$.
- Define the *Complex Multiplication (CM) discriminant* of E to be the square-free part of $4p - t^2$.
- For given square-free $D > 0$, Complex Multiplication (CM) method constructs elliptic curve with CM discriminant D.
  - Used to construct curves with specified number of points.

Effectiveness of the CM Method

- Running time of CM method depends on the class number h_D of $\mathbb{Q}(\sqrt{-D})$.
  - Bottleneck is computing the *Hilbert class polynomial*, a polynomial of degree h_D.
  - Best known algorithms run in (roughly) $O(h_D^2) = O(D)$ (Enge).
- Can be efficiently implemented if h_D not too large.
  - Current record is $h_D = 10^5$.
  - Equivalent to $D \approx 10^{10}$.

Existence Conditions for Pairing-Friendly Curves

- Fix an embedding degree k, and look for parameters $t =$ trace, $r =$ subgroup size, $p =$ field size, satisfying:
  1. p and r are prime.
  2. r divides $p + 1 - t$.
     - $E(\mathbb{F}_p)$ has a point of order r.
  3. r divides $\Phi_k(p)$, where Φ_k is the kth cyclotomic polynomial.
     - p has exact order k in $(\mathbb{Z}/r\mathbb{Z})^\times$.
  4. $4p - t^2 = Dy^2$ for some sufficiently small D and some $y \in \mathbb{Z}$. (This is the "CM equation.")
- For such t, r, p, if D is not too large ($\sim 10^{10}$) we can construct an elliptic curve E over $\mathbb{F}_p$ with an order-r subgroup and embedding degree k.

Observations about the CM Method

- The embedding degree condition $r \mid \Phi_k(p)$ can be replaced with $r \mid \Phi_k(t-1)$.
  - r divides $p+1-t$ implies $p \equiv t-1 \pmod{r}$.
- We can use $\#E(\mathbb{F}_p) = p+1-t$ to write the "CM equation" in two ways:

$$\begin{aligned} Dy^2 &= 4p - t^2 \\ Dy^2 &= 4hr - (t-2)^2. \end{aligned}$$

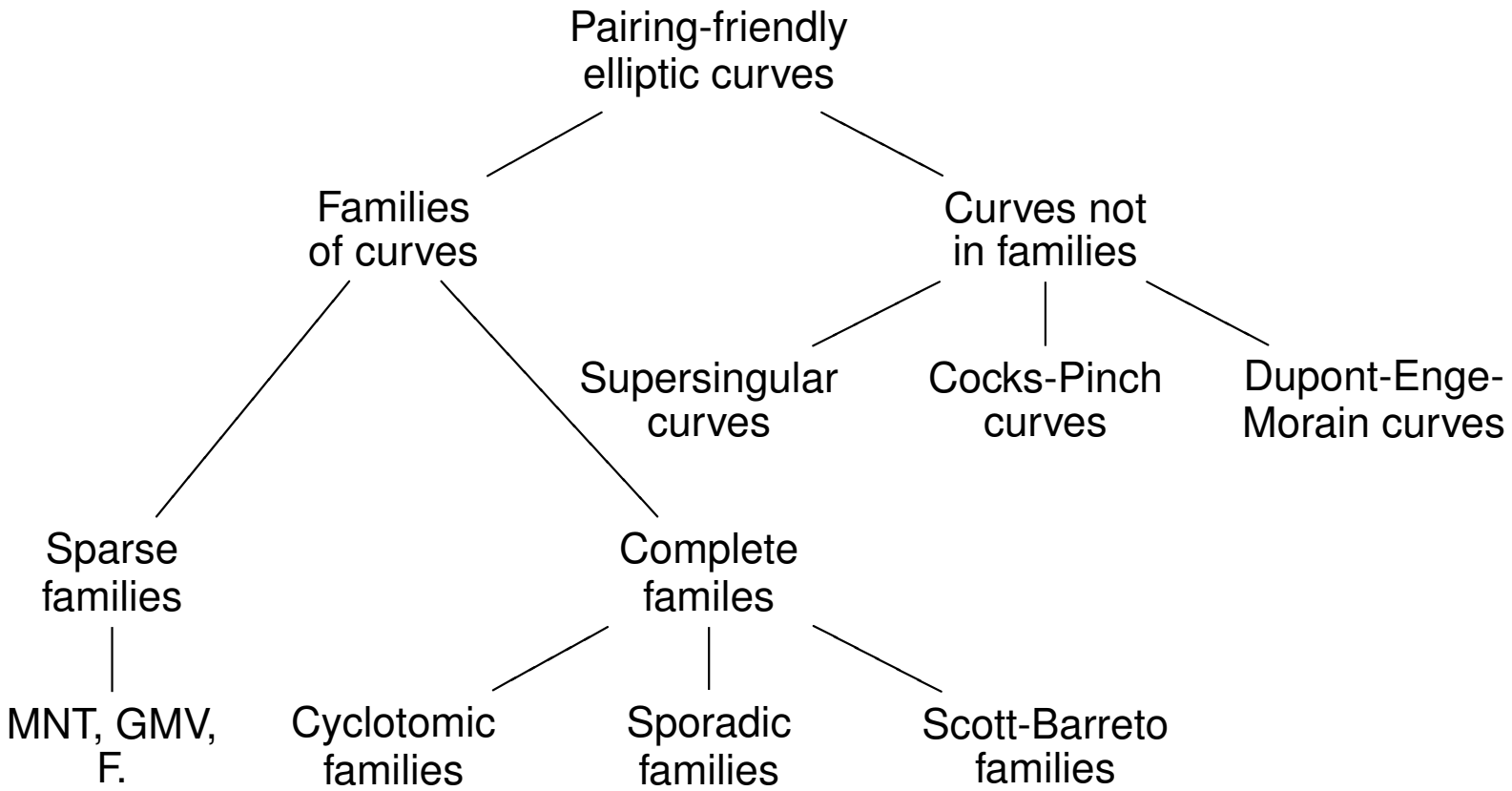
  - h is a "cofactor" satisfying $\#E(\mathbb{F}_p) = hr$.
  - Set $h = 1$ if we want $\#E(\mathbb{F}_p)$ to be prime.

Outline

Our Classification

1. Curves not in families: Construction gives t, r, p directly; repeat construction to get different curve parameters.
2. Families of curves: Parametrize t, r, p as polynomials $t(x), r(x), p(x)$; plug in x to get curve parameters.
   1. Sparse families: Solutions (x, y) to $4p(x) - t(x)^2 = Dy^2$ grow exponentially.
   2. Complete families: Solutions (x, y) to $4p(x) - t(x)^2 = Dy^2$ exist for any x.
      - Further classified by properties of $r(x)$.

Classification of Pairing-Friendly Elliptic Curves



Outline

Overview of the Cocks-Pinch Method

- Recall: for fixed D, k, we are looking for t, r, p satisfying certain divisibility conditions and the CM equation

$$Dy^2 = 4p - t^2.$$

- Cocks-Pinch strategy: Choose r, compute t satisfying divisibility conditions, compute y, p satisfying CM equation.
  - Good for constructing curves with arbitrary k.
  - Can't construct curves of prime order; usually $\rho \approx 2$.
  - Many curves possible, easy to specify bit sizes.
  - Has been generalized to produce families of curves.

The Cocks-Pinch Method

1. Fix D, k, and choose a prime r.
   - Require that k divides $r-1$ and $-D$ is a square mod r.
2. Compute $t' = 1 + x^{(r-1)/k}$ for x a generator of $(\mathbb{Z}/r\mathbb{Z})^\times$.
3. Compute $y' = (t'-2)/\sqrt{-D} \pmod{r}$.
4. Lift t', y' to integers t, y, and compute $p = (t^2 + Dy^2)/4$ (in $\mathbb{Q}$).
5. If p is an integer and prime, use CM method to construct elliptic curve over $\mathbb{F}_p$ with an order-r subgroup.

- y is constructed so that CM equation $Dy^2 = 4p - t^2$ is automatically satisfied.
- Since t', y' are essentially random integers in $[0, r)$, $p \approx r^2$, so $\rho \approx 2$.

Overview of the Dupont-Enge-Morain Method

- Recall: for fixed D, k, we are looking for t, r, p satisfying certain divisibility conditions and the CM equation

$$Dy^2 = 4p - t^2$$

- Dupont-Enge-Morain strategy: Choose D, y, use resultants to find t and r simultaneously, compute p such that CM equation is satisfied.
  - Good for constructing curves with arbitrary k.
  - Can't construct curves of prime order; usually $\rho \approx 2$.
  - Has not been generalized to produce families of curves.

The Dupont-Enge-Morain Method

1. Fix k, choose D, y, compute resultant

$$\mathrm{Res}_t(\Phi_k(t-1), Dy^2 - (t-2)^2).$$

2. If resultant has a large prime factor r, then can compute t' such that $\Phi_k(t-1) \equiv Dy^2 - (t-2) \equiv 0 \pmod{r}$.
3. Lift t' to integer t, compute $p = (t^2 + Dy^2)/4$.
4. If p is an integer and prime, use CM method to construct elliptic curve over $\mathbb{F}_p$ with an order-r subgroup and embedding degree k.

- Since t' is essentially random in $[0, r)$, $p \approx r^2$, so $\rho \approx 2$.

Outline

Overview of the Miyaji-Nakabayashi-Takano Method

- Recall: for fixed D, k, we are looking for t, r, p satisfying certain divisibility conditions and the CM equation

$$Dy^2 = 4p - t^2 = 4hr - (t-2)^2$$

  for some y.
- Idea: Parametrize t, r, p, h as polynomials $t(x), r(x), p(x), h(x)$.
- Miyaji-Nakabayashi-Takano method: Choose $t(x), h(x)$, compute $r(x)$ satisfying divisibility conditions, solve CM equation in 2 variables x, y.
  - Good for constructing curves of prime order.
  - Only 4 possible embedding degrees: $k = 3, 4, 6, 10$.
  - Solutions to CM equation grow exponentially.

The Miyaji-Nakabayshi-Takano (MNT) Method

1. Fix D, k, and choose polynomials $t(x), h(x)$.
   - $h(x) = 1$ if searching for curves of prime order.
2. Choose $r(x)$ an irreducible factor of $\Phi_k(t(x) - 1)$.
3. Compute $p(x) = h(x)r(x) + t(x) - 1$.
4. Find integer solutions (x_0, y_0) to CM equation $Dy^2 = 4h(x)r(x) - (t(x) - 2)^2$.
5. If $p(x_0)$, $r(x_0)$ are both prime for some x_0, use CM method to construct elliptic curve over $\mathbb{F}_{p(x_0)}$ with $h(x_0)r(x_0)$ points.

- For the rest of this section, we will assume $h(x)$ is a constant.

Obstacles to the MNT Method

- Step 4 is the difficult part: finding integer solutions (x_0, y_0) to
$$Dy^2 = 4hr(x) - (t(x) - 2)^2.$$
- If $f(x) = 4hr(x) - (t(x) - 2)^2$ has degree ≥ 3 and no multiple roots, then $Dy^2 = f(x)$ has only a finite number of integer solutions! (Siegel's Theorem)
- Consequence: need to choose $t(x)$, $r(x)$ so that $f(x)$ is quadratic or has multiple roots.
- This is hard to do for $k > 6$, since $\deg r(x)$ must be a multiple of $\deg \Phi_k > 2$.

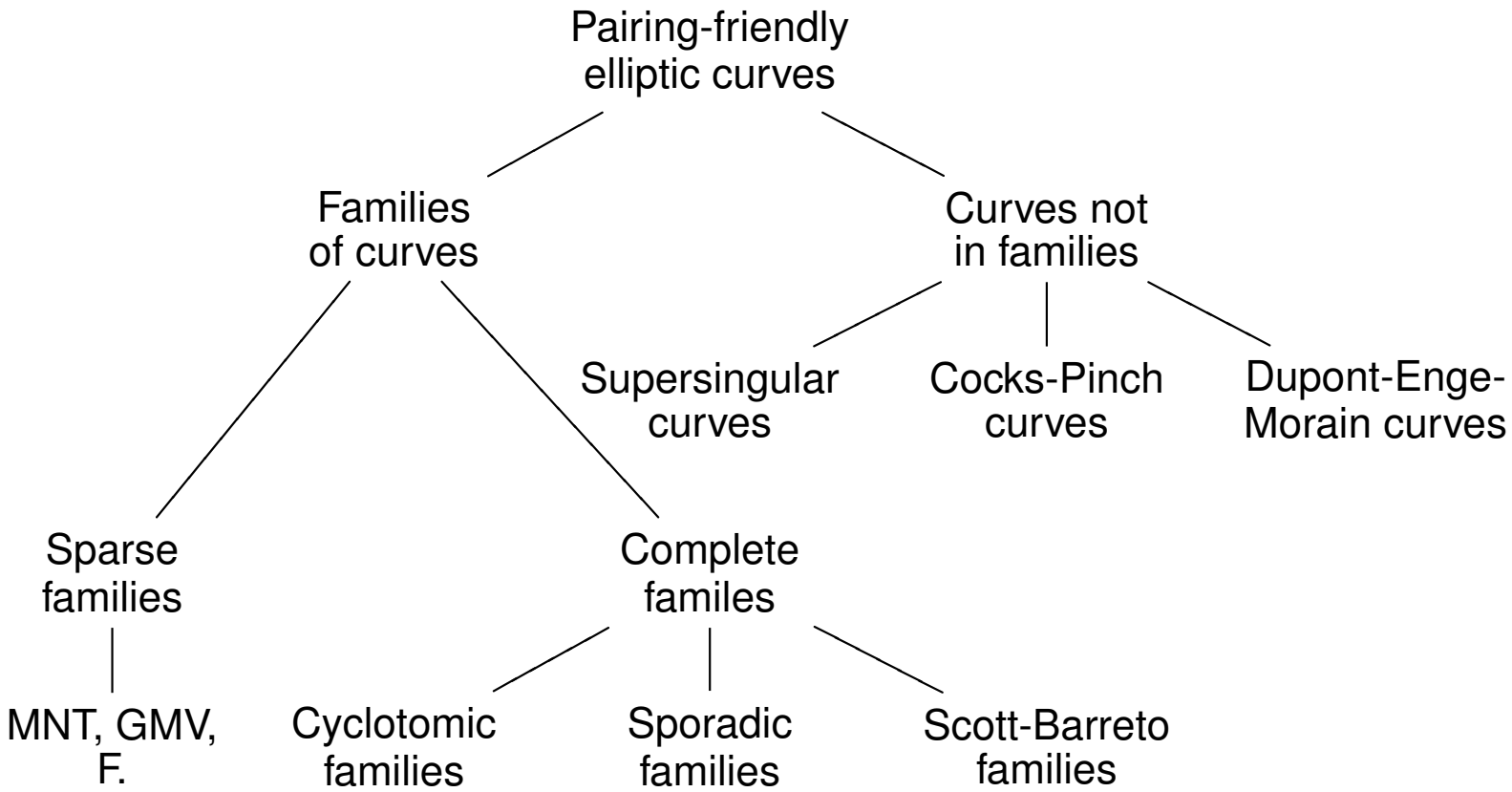
The MNT Solution for $k = 3, 4, 6$

- Goal: Choose $t(x)$, find factor $r(x)$ of $\Phi_k(t(x) - 1)$, such that $f(x) = 4hr(x) - (t(x) - 2)^2$ is quadratic.
- Solution:
  1. Choose $t(x)$ linear; then $r(x)$ is quadratic, and so is $f(x)$.
  2. Use standard algorithms to find solutions (x_0, y_0) to $Dy^2 = f(x)$.
  3. If no solutions of appropriate size, or $p(x)$ or $r(x)$ not prime, choose different D and try again.
- Since construction depends on solving a Pell-like equation, MNT curves of prime order are sparse (Luca-Shparlinski).
- Galbraith-McKee-Valença, Scott-Barreto extend MNT idea by allowing cofactor $h(x) \neq 1$, so that $\#E(\mathbb{F}_p) = h(x)r(x)$.
  - Find many more suitable curves than original MNT construction.

F. Solution for $k = 10$

- Goal: Choose $t(x)$, find factor $r(x)$ of $\Phi_{10}(t(x) - 1)$, such that $f(x) = 4r(x) - (t(x) - 2)^2$ is quadratic.
  - All irred. factors of $\Phi_{10}(t(x) - 1)$ must have $4 \mid$ degree.
- Key observation: Need to choose $r(x)$, $t(x)$ such that the leading terms of $4r$ and t^2 cancel out.
  - Smallest possible case: $\deg r = 4$, $\deg t = 2$.
- Galbraith-McKee-Valença: Characterized quadratic $t(x)$ such that $\Phi_{10}(t(x) - 1)$ factors into two quartics.
- One of these $t(x)$ gives the desired cancellation!
- Construct curves via Pell-like equation as in MNT solution.
  - Like MNT curves, $k = 10$ curves are expected to be sparse.
  - Can't be extended to allow cofactors $h \neq 1$.

Classification of Pairing-Friendly Elliptic Curves



Outline

Overview of Complete Families

- Recall: for fixed D, k, we are looking for t, r, p satisfying certain divisibility conditions and the CM equation

$$Dy^2 = 4p - t^2 = 4hr - (t-2)^2$$

  for some y.
- Idea: Parametrize t, r, p, h as polynomials $t(x), r(x), p(x), h(x)$.
- Complete families method: Choose $r(x)$, use properties of the number field $K = \mathbb{Q}[x]/(r(x))$ to compute $t(x), p(x)$ satisfying CM equation for any x.
  - Good for constructing curves with arbitrary k.
  - Usually gives curves with $1 < \rho < 2$.
  - Easy to specify bit sizes of curves (if k not too large).

Constructing Complete Families of Curves

1. Fix D, k, choose an irreducible polynomial $r(x)$.
   - Let K be the number field $\mathbb{Q}[x]/(r(x))$.
   - Require that K contain a kth root of unity ζ_k.
2. Choose $t(x)$ to be a polynomial representing $1 + \zeta_k \in K$.
3. Compute $y(x)$ so that CM equation is satisfied in K.
4. Compute $p(x) = (t(x)^2 + Dy(x)^2)/4$ (in $\mathbb{Q}[x]$).
5. If $p(x_0)$ is an integer for some x_0 and $p(x_0), r(x_0)$ are prime, use CM method to construct elliptic curve over $\mathbb{F}_{p(x_0)}$ with an order-$r(x_0)$ subgroup and embedding degree k.

Properties of Complete Families

- For large x, $\rho \approx \deg p / \deg r$.
- Working modulo $r(x)$, we can choose $t(x), y(x)$ such that $\deg t, \deg y < \deg r$, so $\deg p \leq 2 \deg r - 2$.
  - Can always get $\rho < 2$, improving on CP, DEM methods.
  - With clever choices of $r(x)$, $t(x)$, ρ can be decreased even further.
  - Best current results: $\rho = \frac{k+1}{k-1}$ for k prime $\equiv 3 \pmod 4$.
- No restrictions on k, and many values of x_0, D produce curves.
  - Compare with sparse families: $k \leq 10$, and values of x_0 grow exponentially.
- Complete families subdivided according to type of number field $K = \mathbb{Q}[x]/(r(x))$.

Cyclotomic Families

- Idea of Barreto-Lynn-Scott, Brezing-Weng: Fix k, D, choose $r(x)$ to be cyclotomic polynomial $\Phi_\ell(x)$ with $k \mid \ell$, $4D \mid \ell$.
  - Then $K = \mathbb{Q}[x]/(r(x)) \cong \mathbb{Q}(\zeta_\ell)$, and K contains $\zeta_k, \sqrt{-D}$.
  - E.g., $k = 8$, $D = 3$, $r(x) = \Phi_{24}(x)$, $K \cong \mathbb{Q}(\zeta_{24})$.
- CM equation $Dy^2 = 4h(x)r(x) - (t(x) - 2)^2$ factors in K as

$$\left(\zeta_k - 1 + y\sqrt{-D}\right)\left(\zeta_k - 1 - y\sqrt{-D}\right) \equiv 0 \pmod{r(x)}.$$

- If $y(x)$ is a polynomial mapping to $(\zeta_k - 1)/\sqrt{-D}$ in K, then CM equation automatically satisfied.
- Restriction: $\sqrt{-D}$ in $\mathbb{Q}(\zeta_\ell)$ implies D divides ℓ.
  - In practice, usually set $D = 1, 2$, or 3.

Example of a Cyclotomic Family (Brezing-Weng)

- Fix $k = 8$, $D = 3$.
- $r(x) = \Phi_{24}(x) = x^8 - x^4 + 1$ defines $K = \mathbb{Q}[x]/(r(x)) \cong \mathbb{Q}(\zeta_{24})$.
- $\zeta_8 \mapsto x^5 - x$ in K, so choose $t(x) = x^5 - x + 1$.
- $\sqrt{-3} \mapsto 2x^4 - 1$ in K, so choose

$$y(x) = \frac{\zeta_8 - 1}{\sqrt{-3}} = \frac{1}{3}(x^5 + 2x^4 + x - 1)$$

- Compute

$$p(x) = \frac{t(x)^2 + Dy(x)^2}{4} = \frac{1}{3}(x-1)^2(x^8 - x^4 + 1) + x^9$$

- Evaluating at $x = 1000726$ gives 160-bit prime r and 198-bit prime p.
  - $y^2 = x^3 + 14$ over $\mathbb{F}_p$ has point of order r, embedding degree 8, and $\rho \approx 5/4$

Sporadic Families

- Idea of Barreto-Naehrig: Fix k, D, choose $r(x)$ so that $K = \mathbb{Q}[x]/(r(x))$ is an *extension* of a cyclotomic field containing $\zeta_k, \sqrt{-D}$.
- How? Choose $u(x)$ so that $\Phi_k(u(x)) = r(x)r'(x)$, set $K = \mathbb{Q}(x)/(r(x))$.
- Construct $t(x), y(x), p(x)$ as with cyclotomic families.

Example: Barreto-Naehrig Curves

- Set $k = 12$, $D = 3$. Look for $u(x)$ such that $\Phi_{12}(u(x))$ factors.
- Galbraith-McKee-Valença: only 2 such quadratic $u(x)$.
- $u(x) = 6x^2$ gives $r(x) = 36x^4 + 36x^3 + 18x^2 + 6x + 1$.
- Compute $t(x) = 6x^2 + 1$,
  $p(x) = 36x^4 + 36x^3 + 24x^2 + 6x + 1$.
- Since $p(x) + 1 - t(x) = r(x)$ (not just divisible by $r(x)$), $r(x)$ is the size of the full group of points.
  - When $p(x_0), r(x_0)$ are both prime for some x_0, E is a curve of prime order and $k = 12$.
- BN family is only family of prime-order curves that is not sparse – easy to specify the bit sizes of p, r.

Scott-Barreto Families

- Scott-Barreto idea: Fix k, choose $r(x)$ so that $K = \mathbb{Q}(x)/(r(x))$ is an extension of a cyclotomic field containing ζ_k and *not containing* $\sqrt{-D}$ for small D.
- Compute $t(x)$, search (via computer) for $h(x)$ that makes right side of CM equation

$$Dy^2 = 4h(x)r(x) - (t(x) - 2)^2$$

  a linear factor times a perfect square.
- Compute D and $y(x)$ such that CM equation is satisfied for any x.
- Method gives families with $1 < \rho < 2$.
  - Known examples have $k \leq 6$.

Summary: Pairing-Friendly Elliptic Curves

1. Curves not in families (Cocks-Pinch, Dupont-Enge-Morain):
   - Good for constructing curves with arbitrary k.
   - Can't construct curves of prime order ($\rho \approx 2$).
   - Many curves possible, easy to specify bit sizes.
2. Sparse families (Miyaji-Nakabayashi-Takano, F.):
   - Good for constructing curves of prime order.
   - Only 4 possible embedding degrees ($k = 3, 4, 6, 10$).
   - Curves are rare.
3. Complete families (Barreto-Lynn-Scott, Brezing-Weng, Scott-Barreto, others):
   - Good for constructing curves with arbitrary k.
   - Usually can't construct curves of prime order ($1 < \rho < 2$).
     - Exception: Barreto-Naehrig curves with $k = 12$.
   - Many curves possible, easy to specify bit sizes.

The State of the Art

Smallest known ρ-values for families with even embedding degrees k.

k	ρ	Type	k	ρ	Type
4	1	Sparse	22	13/10	Cyclotomic
6	1	Sparse	24	5/4	Cyclotomic
8	5/4	Cyclotomic	26	7/6	Cyclotomic
10	1	Sparse	28	4/3	Cyclotomic
12	1	Sporadic	30	3/2	Cyclotomic
14	4/3	Cyclotomic	32	17/16	Cyclotomic
16	5/4	Cyclotomic	34	9/8	Cyclotomic
18	4/3	Cyclotomic	36	17/12	Cyclotomic
20	11/8	Cyclotomic	38	7/6	Cyclotomic