CS 369P: Polyhedral techniques in combinatorial optimization
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1 Matroid Union

Definition 1 *Given matroids* $\mathcal{M}_1 = (E_1, \mathcal{I}_1), \ldots, \mathcal{M}_k = (E_k, \mathcal{I}_k)$ *then* $\mathcal{M}_1 \vee \mathcal{M}_2 \vee \ldots \vee \mathcal{M}_k$ *is defined on ground set* $E_1 \cup E_2 \cup \ldots \cup E_k$ *so that* $\mathcal{I} = \{I_1 \cup \ldots \cup I_k : I_j \in \mathcal{I}_j\}$.

Remarks:

- Unlike matroid intersection, matroid union generates a matroid.
- Union applies to the independent sets and not the matroids as a whole, i.e. it is an operation on a different level than matroid intersection.
- $E_1, E_2, \ldots, E_k$ need not be disjoint; however, it is interesting even for disjoint sets.
- Despite the differences, matroid intersection and matroid union are closely related.

The proof that this operations generates a matroid relies on the following basic lemma.

Lemma 2 *Given a matroid* $\mathcal{M}' = (E', \mathcal{I}')$ *with a rank function* r'*, and a function* $f : E' \to E$*,*

$$\mathcal{I} = \{f(I') : I' \in \mathcal{I}'\}$$

defines a matroid with rank function

$$r(S) = \min_{T \subseteq S} (|S \setminus T| + r'(f^{-1}(T))).$$

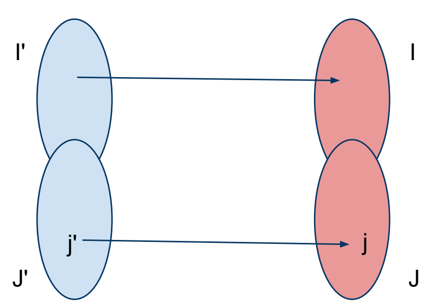

Figure 1: Extension axiom under mapping.

Proof: $\mathcal{I}$ is closed under subsets. We will prove the extension axiom by induction. Let $I, J \in \mathcal{I}$, where $|I| < |J|$. Choose $I', J' \in \mathcal{I}'$, $f(I') = I, f(J') = J, |I'| = |I|, |J'| = |J|$, and $|I' \cap J'|$ as large as possible. Since $\mathcal{M}'$ is a matroid, $\exists j' \in J' \setminus I'$ such that $I' + j' \in \mathcal{I}'$. If $f(j') = j \in I = f(I')$, then j has another pre-image i in I' (since $f(I') = I$). Note that $i \notin J'$ because f is a bijection on J'. However, then we could take $I'' = I - i + j'$ instead of I', thus increasing $|I' \cap J'|$. Hence, $j \in J \setminus I$ proves the extension axiom.

For the rank formula, consider $\mathcal{P}$ = partition matroid with parts $f^{-1}(s)$, $s \in S$. $I \subseteq S$ is independent in $\mathcal{M} \iff \exists I' \in \mathcal{I}', f(I') = I$. Wlog, $|I'| = |I|$, i.e. I' is also independent in $\mathcal{P}$. By the matroid intersection formula,

$$\max_{I \subseteq S, I \in \mathcal{I}} |I| = \max_{I' \in \mathcal{I}' \cap \mathcal{P}} |I'| = \min_{T' \subseteq f^{-1}(S)} (r'(f^{-1}(S) \setminus T') + r_{\mathcal{P}}(T')) = \min_{T \subseteq S} (r'(f^{-1}(S \setminus T)) + |T|).$$

Note that we replaced the minimization over $T' \subseteq f^{-1}(S)$ by minimization over pre-images of $T \subseteq S$; this does not change the minimum, because if T' contains some element of $f^{-1}(s)$, we might as well include all of $f^{-1}(s)$ in T' which does not increase the rank $r_{\mathcal{P}}(T')$ but might decrease the rank $r'(f^{-1}(S) \setminus T')$. $\square$

Theorem 3 *The result of matroid union is a matroid.*

Proof: Assume $E'_1, E'_2, \ldots, E'_k$ disjoint, then $\mathcal{I}' = \{I'_1 \cup I'_2 \cup \ldots \cup I'_k \colon I'_j \text{ independent in } \mathcal{M}'_i\}$. This is clearly a matroid. (Check the axioms; each part E'_i works independently here.)

If $E_1, E_2, \ldots, E_k$ are not disjoint, we can define formally disjoint copies $E'_1, E'_2, \ldots, E'_k$ and a mapping $f \colon \bigcup E'_i \to \bigcup E_i$ which takes each copy E'_i to E_i by a natural bijection. Thus by the previous lemma, $\mathcal{I} = \{f(I') \colon I' \in \mathcal{I}'\}$ is a matroid and this is equivalent to $\mathcal{I} = \{I_1 \cup \ldots \cup I_k \colon I_j \in \mathcal{I}_j\}$.

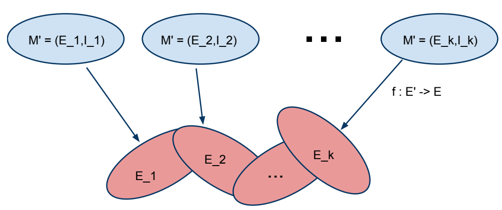

Figure 2: Matroid union from the mapping lemma.

$\square$

Corollary 4 *The rank function of $\mathcal{M}_1 \vee \mathcal{M}_2 \vee \ldots \vee \mathcal{M}_k$ is*

$$r(S) = \min_{T \subseteq S} (|S \setminus T| + r_1(T \cap E_1) + \ldots + r_k(T \cap E_k)).$$

Proof: For disjoint ground sets $E_1, \ldots, E_k$, the rank function is clearly $r(S) = \sum_{i=1}^{k} r_i(S \cap E_i)$. In the general case, by the construction from disjoint ground sets and Lemma 2, we obtain

$$r(S) = \min_{T \subseteq S} (|S \setminus T| + r'(f^{-1}(T))) = \min_{T \subseteq S} (|S \setminus T| + \sum_{i=1}^{k} r_i(T \cap E_i)).$$

$\square$

Algorithms for checking independence in $\mathcal{M}_1 \vee \mathcal{M}_2 \vee \ldots \vee \mathcal{M}_k$ and optimizing over it can be derived from matroid intersection.

2 Applications of matroid union

Theorem 5 *The maximum size of a union of k independent sets in $\mathcal{M}$ is*

$$\min_{S \subseteq E} [|E \setminus S| + k \cdot r(S)].$$

Proof: Directly from matroid union with $\mathcal{M}_1 = \mathcal{M}_2 = \ldots = \mathcal{M}_k$. □

Theorem 6 *The ground set of a matroid $\mathcal{M}$ can be partitioned into k independent sets iff*

$$\forall S \subseteq E; r(S) \geq \frac{1}{k}|S|.$$

Proof: E is independent in $\mathcal{M} \vee \mathcal{M} \vee \ldots \vee \mathcal{M}$ iff $\min_{S \subseteq E}[|E \setminus S| + kr(S)] \geq |E|$. □

Theorem 7 *$\mathcal{M}$ contains k disjoint bases iff*

$$\forall S \subseteq E; k(r(E) - r(S)) \leq |E \setminus S|.$$

Proof: E contains k disjoint bases if its rank in $\mathcal{M} \vee \mathcal{M} \vee \ldots \vee \mathcal{M}$ is at least $kr(E)$ i.e. if $\min_{S \subseteq E}[|E \setminus S| + kr(S)] \geq kr(E)$. □