**CS 369P: Polyhedral techniques in combinatorial optimization**
**Instructor: Jan Vondrák**
**Lecture date: October 5, 2010** **Scribe: Michael Kapralov**

# 1 The matching polytope

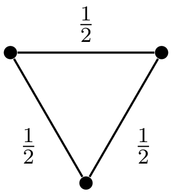
Today, we are interested in the polyhedral description of

$$P_{\text{match}}(G) = \text{conv}\{\chi_M : M \text{ matching in } G\}$$

As we have seen, the description

$$\{\mathbf{x} \geq 0 : \forall v \in V, x(\delta(v)) \leq 1\}$$

is not correct for non-bipartite graphs: For example, a valid constraint missing in this description is



$$\sum_{e \in \text{triangle}} x_e \leq 1$$

since if $\mathbf{x} = \sum_M \alpha_M \chi_M$, then $\sum_{e \in \text{triangle}} x_e = \sum_M \alpha_M |M \cap \text{triangle}| \leq 1$. More generally, for any odd-size set $U \subseteq V$,

$$\sum_{e \in E[U]} x_e \leq \left\lfloor \frac{1}{2}|U| \right\rfloor .$$

We will prove

**Theorem 1** *(Edmonds) The matching polytope of G is given by*

$$P_{matching}(G) = \left\{ \mathbf{x} \geq 0 : \forall v \in V, x(\delta(v)) \leq 1, \; \forall U \subseteq V, |U| = \; odd, x(E(U)) \leq \left\lfloor \frac{1}{2}|U| \right\rfloor \right\} .$$

Note that the number of constraints is exponential in the size of the graph; however, the description will be still useful for us. But first, let us consider the perfect matching polytope.

## 2 The perfect matching polytope

Define

$$P_{\text{perf-match}}(G) = \text{conv}\{\chi_M : M \text{ a perfect matching in } G\}$$

In the bipartite case $P_{\text{perf-match}}(G)$ is given by $\{\mathbf{x} \geq 0 : \forall v \in V; x(\delta(v)) = 1\}$. Again, we need to add odd-set constraints in the bipartite case.

**Theorem 2** *(Edmonds)*

$$\begin{aligned} P_{\textit{perf-match}}(G) &= \{\mathbf{x} \geq 0 : \\ &\quad \forall v \in V; x(\delta(v)) = 1, \\ &\quad \forall U \subseteq V, |U| = \textit{ odd }; x(\delta(U)) \geq 1\}. \end{aligned} \tag{1}$$

**Proof:** Let

$$Q = \{x \geq 0 : \forall v \in V, x(\delta(v)) = 1 \text{ and } \forall U \subseteq V \text{ s.t. } |U| \text{ is odd } x(\delta(U)) \geq 1\}.$$

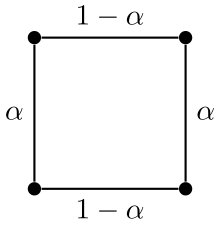
Since every perfect matching in $G$ satisfies the constraints in $Q$, we have that $P_{\text{perf-match}} \subseteq Q$. We now prove that $Q \subseteq P_{\text{perf-match}}(G)$. We proceed by induction on $|E|$.

**Base case.** $|E| = 1$, we must have $x_e = 1$; trivial.

**Inductive step.** Consider a vertex $\mathbf{x} \in Q$. If we prove that $\mathbf{x} \in P_{\text{perf-match}}(G)$ for each vertex of $Q$, it will imply that $Q \subseteq P_{\text{perf-match}}(G)$, since $Q$ is a bounded polytope and hence a convex hull of its vertices.

If $x_e = 0$ for some edge $e$, we can remove the edge and use the inductive hypothesis. If $x_e = 1$ for some edge $e$, then we can remove $e$ together with its endpoints and use the inductive hypothesis.

Suppose that $0 < x_e < 1$ for all $e \in E(G)$. Note that the constraints of $Q$ imply that the degree of every vertex is at least 2. First suppose that all degrees are exactly 2. Then $E$ is a union of cycles, and the values $x_e$ on each cycle alternate between $1 - \alpha$ and $\alpha$ for some $\alpha \in (0, 1)$:



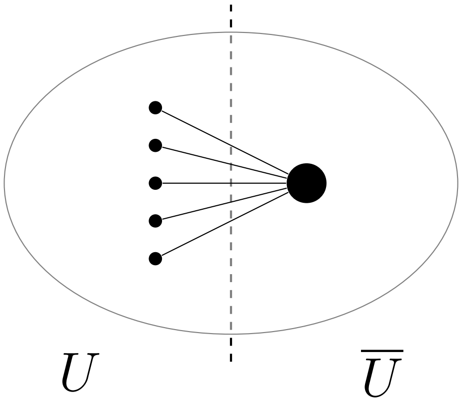
This, however, is not a vertex (consider adding $\epsilon$ to every odd edge and subtracting $\epsilon$ from every even edge – this can be done for sufficiently small positive and negative $\epsilon$).

Thus, we can assume that some degrees are $> 2$, and all are $\geq 2$. Then we have $|E| > |V|$. Since we are considering a vertex of $Q$, this implies that there are $|E| > |V|$ linearly independent tight constraints. We have only $|V|$ vertex-degree constraints; hence there exists $U \subseteq V$, $|U| \geq 3$ odd, such that $x(\delta(U)) = 1$. Let $\overline{U} = V \setminus U$.

Let $G/U$ be the graph obtained by contracting $U$ to a new node, and $G/\overline{U}$ be the graph obtained by contracting $\overline{U}$ to a new node. Define $\mathbf{x}' \in \mathbb{R}^{E(G/U)}, \mathbf{x}'' \in \mathbb{R}^{E(G/\overline{U})}$ as restrictions of $\mathbf{x}$ to the edges that did not disappear in the contraction. Since $\delta(U) = \delta(\overline{U}) = 1$,

**Claim 3** $\mathbf{x}' \in Q(G/U), \mathbf{x}'' \in Q(G/\overline{U})$

**Proof:** Vertex-degree constraints $x(\delta(v)) = 1$ are satisfied because we have $x(\delta(U)) = x(\delta(\overline{U})) = 1$, and the odd set constraints are propagated to to the graph because we shrunk an odd number of vertices into one. $\square$



$U$ was an odd set of size at least 3, so the number of edges in $G/U$ (resp. $G/\overline{U}$) is smaller than in $G$, By the inductive hypothesis, $\mathbf{x}' \in P_{\text{perf-match}}(G/U), \mathbf{x}'' \in P_{\text{perf-match}}(G/\overline{U})$. Thus,

$$\mathbf{x}' = \sum_{M' \text{ perf. matching in } G/U} \alpha_{M'} \chi_{M'},$$
$$\mathbf{x}'' = \sum_{M'' \text{ perf. matching in } G/\overline{U}} \alpha_{M''} \chi_{M''}.$$

Note that $\mathbf{x}$ was a vertex of $Q$, hence a rational point, and consequently $\mathbf{x}', \mathbf{x}''$ are rational as well. By choosing a common denominator, we can write

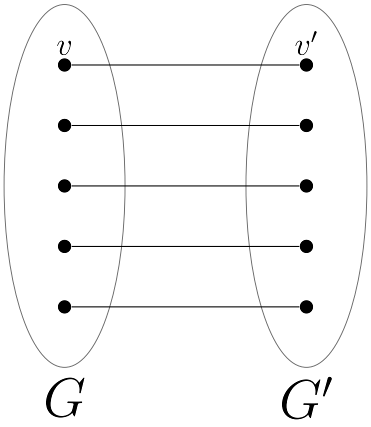
$$\mathbf{x}' = \frac{1}{N} \sum_{i=1}^{N} \chi_{M_i'},$$
$$\mathbf{x}'' = \frac{1}{N} \sum_{i=1}^{N} \chi_{M_i''},$$

where $M_i'$ (resp. $M_i''$) are matchings in $G/U$ (resp. $G/\overline{U}$). Since $\mathbf{x}'$ and $\mathbf{x}''$ agree on $\delta(U)$, we can pair up the matchings $M_i'$ and $M_i''$ so that the matchings in each pair $(M_i', M_i'')$ use the same edge across $(U, \overline{U})$. This implies that $M_i = M_i' \cup M_i''$ is a matching in $G$. Thus,

$$\mathbf{x} = \frac{1}{N} \sum_{i=1}^{N} \chi_{M_i}.$$

$\square$

As a consequence, we can derive Theorem 1.



**Proof:**(Theorem 1) Let $G'$ be a copy of $G$. Let $\tilde{G} = G + G' + \bigcup_{v \in G}(v, v')$, as shown in the figure.

Define $\tilde{\mathbf{x}}$ on the edges of $\tilde{G}$ as $\tilde{x}_e = \tilde{x}_{e'} = x_e$, and $\tilde{x}_{(v,v')} = 1 - x(\delta(v)), \forall v \in V(G)$. If we prove that $\tilde{\mathbf{x}} \in P_{\text{perf-match}}(\tilde{G})$, then we are finished because if $\tilde{\mathbf{x}}$ decomposes into a convex combination of perfect matchings in $\tilde{G}$, by restricting to $G$ we obtain a convex combination of matchings in $G$ that sums up to $\mathbf{x}$. $\square$

**Claim 4** $\tilde{\mathbf{x}} \in P_{\textit{perf-match}}(\tilde{G})$.

**Proof:** Denote $\tilde{\delta}(v) = \{\text{edges in } \tilde{G} \text{ incident to } v\}$. First, we have $\tilde{x}(\tilde{\delta}(v)) = 1$ for all $v \in V(\tilde{G})$. We need to prove that $\forall \tilde{U}$ odd in $\tilde{G}$, one has $\tilde{x}(\tilde{\delta}(\tilde{U}))) \geq 1$. Consider such $\tilde{U}$ of odd cardinality. Let $\tilde{U} = X \cup Y'$, where $X \subseteq V(G), Y' \subseteq V(G')$ (for any set $S \subseteq V(G)$ we define the corresponding set in $G'$ by $S'$). Since $|\tilde{U}|$ is odd, we have that either $|X \setminus Y|$ or $|Y \setminus X|$ is odd, wlog assume that $|X \setminus Y|$ is odd.

Using that $\mathbf{x}$ satisfies $x(E[X \setminus Y]) \leq \lfloor \frac{1}{2}|X \setminus Y| \rfloor = \frac{1}{2}|X \setminus Y| - \frac{1}{2}$, we get

$$\tilde{x}(\tilde{\delta}(X \setminus Y)) = \sum_{v \in X \setminus Y} \tilde{x}(\tilde{\delta}(v)) - 2\tilde{x}(E[X \setminus Y]) \geq |X \setminus Y| - 2 \left\lfloor \frac{1}{2}|X \setminus Y| \right\rfloor = 1.$$

Finally,

$$\tilde{x}(\tilde{\delta}(X \cup Y')) \geq \tilde{x}(\tilde{\delta}(X \setminus Y)) \geq 1,$$

where we used the fact that edges that go from $X \setminus Y$ to $X \cap Y$ are counted in $\tilde{x}(\tilde{\delta}(X \setminus Y'))$ as edges from $X' \cap Y'$ to $X' \setminus Y'$ (see the last figure). So $\tilde{\mathbf{x}}$ satisfies the constraints of $P_{\text{perf-match}}(\tilde{G})$. $\square$