**CS 369P: Polyhedral techniques in combinatorial optimization**
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# 1 Weighted non-bipartite matching

Today we extend Edmond's matching algorithm to weighted graphs. The minimum weight perfect matching problem can be written as the following linear program:

$$\begin{array}{lll} \min & \sum_{e\in E} w_e x_e & \\ \text{s.t.} & \forall v \in V & x(\delta(v)) = 1 \\ & \forall U \subset V,\ |U| = \text{odd} & x(\delta(U)) \geq 1 \\ & \forall e \in E & x_e \geq 0 \end{array}$$

But this program has exponentially-many constraints.

One approach would be to use the ellipsoid algorithm: if we can implement a "separation oracle" in polynomial time, then we can solve the LP. The separation problem is, given $\mathbf{x}$, check all the constraints. Padberg and Rao showed how to solve the problem of minimizing $x(\delta(U))$ over all odd $U$. If the answer is at least 1, then the constraints are satisfied, and if for some $U$ $x(\delta(U)) < 1$, then we have a separating hyperplane. However, we will present a different, more direct algorithm here.

## 1.1 Primal-dual algorithms

We use a primal-dual algorithm due to Edmonds. A primal-dual algorithm is a combinatorial algorithm that solves an LP using the primal/dual structure and was a major breakthrough in algorithm design at the time. The general approach is to keep track of a primal solution $\mathbf{x}$ and its set of tight constraints and try to find a dual solution $\mathbf{y}$ that certifies $\mathbf{x}$ is optimal. If this is impossible, the dual solution will give us a way to improve $\mathbf{x}$.

More specifically, for the following primal and dual LPs

$$\begin{array}{lll} \min & \mathbf{c}^T\mathbf{x} & \\ \text{s.t.} & \mathbf{A}\mathbf{x} & \geq \mathbf{b} \\ & \mathbf{x} & \geq 0 \end{array} \qquad\qquad \begin{array}{lll} \max & \mathbf{b}^T\mathbf{y} & \\ \text{s.t.} & \mathbf{A}^T\mathbf{y} & \leq \mathbf{c} \\ & \mathbf{y} & \geq 0 \end{array}$$

we will maintain a solution pair $(\mathbf{x}, \mathbf{y})$, where $\mathbf{y}$ is dual feasible and $x_i > 0$ only if $(\mathbf{A}^T\mathbf{y})_i = c_i$. If $\mathbf{x}$ becomes feasible and $(\mathbf{A}\mathbf{x})_j = b_j$ for $y_j > 0$, then $(\mathbf{x}, \mathbf{y})$ are optimal.

If there is no such $\mathbf{x}$, then by duality there exists $\mathbf{z}$ such that

$$\begin{array}{rll} \mathbf{A}'^T\mathbf{z} & \leq 0 & \text{for all tight constraints in } \mathbf{y} \\ \mathbf{z}_T & \geq 0 & \text{for } T = \{j : y_j = 0\} \\ \mathbf{b}^T\mathbf{z} & > 0 & \end{array}$$

in which case we can move along $\mathbf{z}$ and improve the dual.

## 1.2 Primal-dual algorithm for weighted perfect matching

The dual of the weighted perfect matching LP above is

$$\begin{array}{rlrl} \max & \sum_{|U| \text{ odd}} \pi_U & & \\ \text{s.t.} & \forall e \in E & \sum_{U: e \in \delta(U)} \pi_U & \leq w_e \\ & \forall |U| \text{ odd}, |U| \geq 3 & \pi_U & \geq 0 \end{array}$$

Our algorithm will maintain a feasible dual solution $\pi$ and an infeasible primal solution $M$. Additionally,

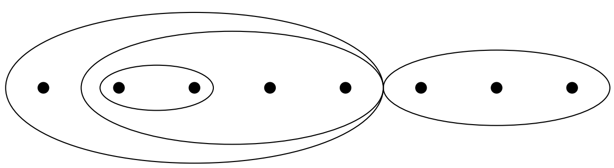
1. Our primal solution satisfies $x_e \in \{0,1\}$, and edges in $M$ always have tight dual constraints, i.e.

$$x_e = 1 \Rightarrow \sum_{U: e \in \delta(U)} \pi_U = w_e$$

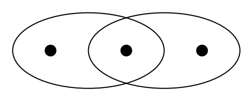
   since we are trying to satisfy complementary slackness.

2. The dual solution uses only sets from some laminar family $\Omega$.

   **Definition 1** *A family of sets $\Omega$ is a* laminar family *if for all $A, B \in \Omega$, $A \cap B = \emptyset$, $A \subseteq B$, or $B \subseteq A$.*



Laminar family



Non-laminar family

   All singletons are part of $\Omega$, and each $U \in \Omega$ with $|U| \geq 3$ corresponds to some contracted blossom, which the algorithm will keep track of, on the maximal sets contained in $U$. Note that $\Omega$ contains $O(n)$ sets, since the sets form a tree with internal degrees at least 3 and $n$ leaves corresponding to vertices.

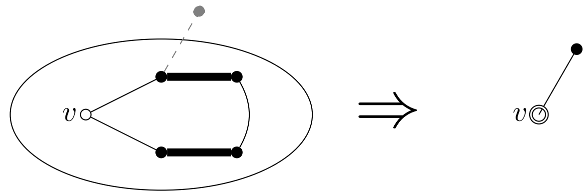
## 1.3 Algorithm

We assume $G$ is simple, has a perfect matching and that $\mathbf{w} \geq 0$. The algorithm is as follows:

0. Initialization: $M = \emptyset$, $\pi = 0$, $\Omega = \{\{v\} \mid v \in V\}$.

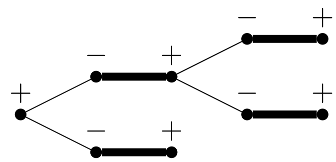
1. Let $X$ be the exposed vertices in $M$ and $E_\pi = \left\{ e \mid \sum_{U: e \in \delta(U)} = w_e \right\}$ be the tight edges. Build alternating trees from $X$ in $E_\pi$. If there is an $M$-augmenting path, extend $M$ and repeat. If $M$ is a perfect matching then stop.

2. If there is an edge forming a blossom—an Even-Even edge—then shrink the blossom and add its vertex set $U$ to $\Omega$ with $\pi_U = 0$. Note that $U$ becomes an Even vertex.




Extend the tree and continue growing the tree or shrinking blossoms as long as possible. If there is an augmenting path, return to 1 (but keep the contracted blossom).

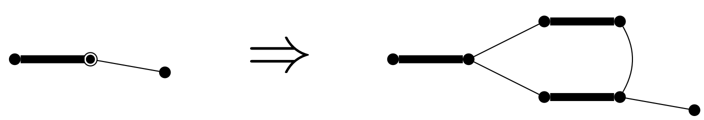
3. If no tree can be extended and there is no Even-Even edge, modify the dual solution:




Increase $\pi_U$ for all Even nodes $U$ and decrease $\pi_U$ for all Odd nodes $U$ at the same rate until you get stuck. Note that $\sum_{|U| \text{ odd}} \pi_U$ increases and edges in the matching remain tight. Some edges might lose tightness, and Even-Even or Even-outside edges might become tight. If the process never gets stuck, then the primal is not feasible.

4. Why did we stop?

   (a) If $\pi_U = 0$ for some $|U| \geq 3$ (singletons can go below 0), de-shrink the blossom, remove $U$ from $\Omega$, and go back to 2. Note $U$ must have been an Odd vertex.




   (b) If a new constraint $\sum_{U: e \in \delta(U)} \pi_U \leq w_e$ becomes tight, add $e$ to $E_\pi$, and go back to 2.

### 1.4 Analysis

**Lemma 2** *If the algorithm terminates, then $M$ is optimal.*

**Proof:** First note that the algorithm only terminates when $M$ is a perfect matching. All edges in $M$ have tight dual constraints, since the algorithm only uses tight edges ($M \subseteq E_\pi$). Now suppose $\pi_U \neq 0$. If $U$ had no edges coming out, there would be an exposed vertex, and the algorithm would not have terminated. Variable $\pi_U$ can only be nonzero if $U$ is a shrunk blossom, and we only shrink sets with at most one edge coming out. In order for a second edge in $\delta(U)$ to be added, $U$ must be expanded, which only occurs when $\pi_U$ drops to 0. Therefore for every nonzero $\pi_U$, $M$ has exactly one edge coming out of $U$. Since $M$ and $\pi$ satisfy complementary slackness they must be optimal solutions. □

Why does the algorithm terminate in polynomial time?

**Lemma 3** *Each set $U$ added to $\Omega$ cannot be removed until the next augmentation of $M$.*

**Proof:** When a set $U$ is shrunk it becomes an Even vertex, so $\pi_U$ can only be increased. Deshrinking other sets does not change its parity. If $U$ is swallowed by a larger set $U'$, $\pi_U$ will not change anymore, and we can argue about $U'$ instead. The only way $U$ can become Odd is by augmenting $M$, and $U$ cannot leave $\Omega$ until it becomes Odd. □

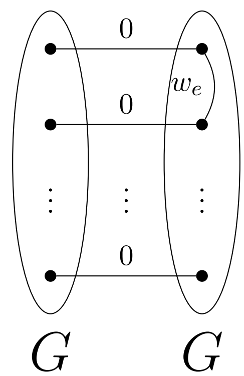
**Corollary 4** *The algorithm terminates after $O(n^2)$ "iterations".*

**Proof:** By Lemma 3, the number of shrinkings between two augmentations of $M$ can be at most the size of $\Omega$ at the beginning of this stage, and similarly the number of deshrinkings between two augmentations of $M$ can be at most $|\Omega|$ at the end of this stage. Since $\Omega$ is laminar, both quantities are $O(n)$ and the number of shrinkings/deshrinkings between two augmentations is $O(n)$. The number of extensions of $E_\pi$ is also $O(n)$: either we add new vertex to some tree, or we form a new blossom, which will be shrunk. Therefore the number of iterations between two augmentations of $M$ is $O(n)$, and $M$ will be augmented exactly $n/2$ times. □

This proves that minimum-weight perfect matching can be found in polynomial time. Similarly, maximum-weight perfect matching can be found in polynomial time, by flipping the weights. Finally, we can also solve the maximum-weight matching problem.

**Corollary 5** *Maximum-weight matching can be found in polynomial time.*

**Proof:** We reduce it to max-weight perfect matching. Create two copies of the graph $G$, with corresponding nodes in each graph connected by edges of weight 0. The max-weight perfect matching in the resulting graph is twice the max-weight matching in $G$.

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