CS 369P: Polyhedral techniques in combinatorial optimization
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1 The greedy algorithm for matroids

The following algorithm finds the maximum weight base in a matroid $\mathcal{M} = (E, \mathcal{I})$

Algorithm 1 Greedy algorithm for selecting the max-weight base of a matroid

Input: a matroid $\mathcal{M} = (E, \mathcal{I})$, where $E = \{1, 2, \ldots, n\}$ is the ground set, and weight of i is w_i.
Output: A base $B \in \mathcal{I}$ such that $w(B) = \max_{B \in \mathcal{B}} w(B)$.

Relabel the elements of the matroid so that $w_1 \geq w_2 \geq \ldots \geq w_n$.
$S \leftarrow \emptyset$.
for $i \leftarrow 1$ to n **do**
  if $S + i \in \mathcal{I}$ **then**
    $S \leftarrow S + i$.
  end if
end for
return S

Theorem 1 (Rado/Gale) *For any ground set $E = \{1, 2, \ldots, n\}$, and a family of subsets $\mathcal{I} \subset 2^E$, Algorithm 1 returns the maximum-weight base for any set of weights $w : E \to \mathbb{R}$ if and only if $\mathcal{M} = (E, \mathcal{I})$ is a matroid.*

We prove this theorem in two parts.

Claim 2 *($\Leftarrow$ part) Suppose that $(E, \mathcal{I})$ is a matroid. For any set of weights assigned to the elements of E, Algorithm 1 returns the maximum-weight base.*

Proof: Wlog assume that $w_1 \geq w_2 \geq \ldots \geq w_n$. We prove that at any point of the execution of the algorithm, there exists an optimal base B such that $S \subseteq B$ and $B \setminus S$ is among the remaining elements.

In particular, let S_i be the set S after observing the first i elements (e.g. $S_0 = \emptyset$). We use induction to show that for any S_i, there exists an optimal base B_i such that $S \subseteq B_i$, and $B_i \setminus S_i \subset \{i+1, \ldots, n\}$. Note this certainly implies the claim, since we get $S_n = B_n$ is an optimal base. The base case of the induction is trivially satisfied for $S_0 = \emptyset$.

Suppose before the i-th iteration we have $S_{i-1} \subset B_{i-1}$ where B_{i-1} is a max-weight base. If the algorithm does not add i to S_{i-1}, it means that $S_{i-1} + i \notin \mathcal{I}$ and therefore i cannot be in B_{i-1} (by the downward closed-ness property of the matroid). Therefore, $B_i = B_{i-1}$ satisfies the induction statement.

Now assume that the algorithm adds i to S_{i-1}. By the induction hypothesis, if $i \in B_{i-1}$, then $S_i = S_{i-1} + i \subset B_{i-1}$, and we can set $B_i = B_{i-1}$. Otherwise, by the extension axiom of the matroid $\mathcal{M}$, the set S_i can be extended from B_{i-1} until it becomes a base, say B'. Since $i \notin B_{i-1}$, we have

$B' \setminus B_{i-1} = \{i\}$. Moreover, since B' is also a base, we have $|B_{i-1} \setminus B'| = 1$, so let $B_{i-1} \setminus B' = \{j\}$. Therefore we can write $B' = B_{i-1} + i - j$.

Since $B_{i-1} \setminus S_i \subset \{i+1, \ldots, n\}$, we have $j \in \{i+1, \ldots, n\}$ is one of the remaining elements. Therefore, since the algorithm orders the element decreasingly according to their weights, we have $w_j \leq w_i$. But this means that $w(B') = w(B_{i-1}) + w_i - w_j \geq w(B_{i-1})$. By the optimality assumption of B_{i-1}, we have $w(B') = w(B_{i-1})$, hence $B_i = B'$ satisfies the induction statement. □

Claim 3 *($\Rightarrow$ part) Suppose $(E, \mathcal{I})$ is not a matroid. There exists an assignment of weights to the elements of E such that algorithm 1 does not return a maximum-weight base.*

Proof: If $(E, \mathcal{I})$ is not a matroid, it does not satisfy at least one of the two properties of the matroid. Suppose $\mathcal{I}$ is not a downward-closed family of sets. Therefore, there exist two sets $S \subset T$, $T \in \mathcal{I}$, but $S \notin \mathcal{I}$. Suppose we assign the weights as follows:

$$\forall 1 \leq i \leq n,\ w_i = \begin{cases} 2 & i \in S \\ 1 & i \in T \setminus S \\ 0 & \text{otherwise} \end{cases}$$

By the weight assignment, the algorithm first considers the elements of S, then the elements of T, and then the rest of the elements. The elements in $E \setminus S$ are worth nothing, thus every optimal base must contain T. Suppose the algorithm selects a subset $S_1 \subset S$ after observing the elements of S. Since $S \notin \mathcal{I}$, we have $S_1 \neq S$. Out of the remaining elements, the algorithm can get value at most $|T \setminus S|$. If S_2 is the final set chosen by the algorithm, we have

$$w(S_2) = 2|S_1| + w(S_2 \setminus S) < 2|S| + |T \setminus S| = w(T).$$

Now suppose $(E, \mathcal{I})$ is not a matroid because the extension axiom is violated (assume the downward closed property). In particular, let $S, T \in \mathcal{I}$ be two independent sets such that $|S| < |T|$, and for all $i \in T \setminus S$, $S + i \notin \mathcal{I}$. We use the following weights:

$$\forall 1 \leq i \leq n,\ w_i = \begin{cases} 1 + \frac{1}{2|S|} & i \in S \\ 1 & i \in T \setminus S \\ 0 & \text{otherwise} \end{cases}$$

Note that S is not necessarily a subset of T here. This time, because of the downward closedness property the algorithm would select all of the elements of S. But this means that it can not add any element in $T \setminus S$, as this would violate independence. Further elements do not bring any value anymore, so if S_2 is the solution returned by the algorithm,

$$w(S_2) = w(S) = |S| \left(1 + \frac{1}{2|S|}\right) = |S| + \frac{1}{2}$$

while the value of T is

$$w(T) \geq |T| \geq |S| + 1.$$

□

The following properties can be shown using the above theorem.

1. Let S_i be the set of elements chosen by the algorithm after observing the first i elements. Then S_i is always a base of those i elements.

2. Finding the maximum weight base in a matroid is in fact equivalent to finding the minimum weight base. Let $w_{max} = \max_{1 \le i \le n} w_i$ be the maximum weight assigned to the elements, to find the minimum weight base it is sufficient to replace $w_i := w_{max} - w_i$, for all $i \in E$.

3. Also by considering the same proof, it is straightforward that if the weights are non-negative, then the weight of the maximum weight independent set will be the same as the weight of the maximum weight base. In general, we can say the weight of the maximum weight independent set among the elements with non-negative weights is the same as the weight of the maximum weight base of those elements.

2 The span function in matroids

The following definition of a "span" of a set of elements in a matroid is similar to the span of a set vectors in $\mathbb{R}^n$:

Definition 4 *Let $\mathcal{M} = (E, \mathcal{I})$ be a matroid. For any set $S \subset E$ define*

$$\mathit{span}(S) := \{i \in E : \ \mathit{rank}(S + i) = \mathit{rank}(S)\}.$$

Suppose we assign distinct weight to the elements of the matroid (i.e. $w_i \neq w_j$ for all $i, j \in E$), then the maximum weight base is unique. Using the above definition, we can simply describe the optimal maximum weight base as follows:

Lemma 5 *Let $\mathcal{M} = (E, \mathcal{I})$ be a give matroid, and let $w_1 > w_2 > \ldots > w_n$ be the weights assigned to the elements $\{1, 2, \ldots, n\} = E$. Then, the maximum weight base is equal to:*

$$\mathcal{B}_{opt} := \{i : i \notin \mathit{span}(\{1, \ldots, i-1\})\}.$$

Before proving this lemma, we need to prove some basic properties regarding the span function. In these lemmas $\mathcal{M} = (E, \mathcal{I})$ is always our matroid and S, T are subsets of elements in E.

Lemma 6 *Let $S \subseteq E$. Then, any base of S is also a base of $\mathit{span}(S)$.*

Proof: We prove by contradiction. Let B be a base of S which is not a base of $\text{span}(S)$. Since B is a base of S, we have $|B| = \text{rank}(S)$. Also, since B is not a base of $\text{span}(S)$, it means that there is some element $i \in \text{span}(S) \setminus B$, such that $B + i \in \mathcal{I}$. Therefore,

$$\text{rank}(S + i) \ge \text{rank}(B + i) > \text{rank}(B) = \text{rank}(S).$$

This contradicts the definition of $i \in \text{span}(S)$. □

Lemma 7 *Let $S \subseteq E$. For any element $i \in E \setminus S$, and any base B of S, $i \notin \mathit{span}(S)$ if and only if $B + i \in \mathcal{I}$.*

Proof: If $i \in span(S)$, then by Lemma 6, B is also a base of $span(S)$, and thus a base of $S+i$. Therefore, $B+i \notin \mathcal{I}$.

Conversely, suppose $i \notin span(S)$. Therefore, $rank(S+i) > rank(S)$, and there is an independent set $B'+i$, where $B' \subset S$ and $|B'+i| = rank(S)+1$. In other words, B' is a base of S. Now consider any base B of S. This is also an independent set of $S+i$. Since $|B| < |B'+i|$, by the extension axiom, it can be extended by adding an element from $B'+i$. But that element must be i (otherwise, B was not a base of S), and thus $B+i$ is a base of $S+i$. $\square$

Lemma 8 *Let $S \subseteq T \subseteq E$. For any element $i \in E$, if i is in the span of S, then it is also in the span of T:*

$$rank(S+i) = rank(S) \Rightarrow rank(T+i) = rank(T)$$

Proof: Let B be a base of S, and B_T be a base of T. By the extension axiom, B can be extended to a base B' of T from the elements of B_T (note that $B' \setminus B \subset T \setminus S$). Since $rank(S+i) = rank(S)$, we have $B+i \notin \mathcal{I}$. Therefore, since $B \subset B'$, by the downward closedness axiom, $B'+i \notin \mathcal{I}$ either. By Lemma 7, i should lie in the span of T (i.e. $i \in span(T)$), and hence $rank(T+i) = rank(T)$. $\square$

Corollary 9 *For any $S \subseteq T \subseteq E$, $span(S) \subseteq span(T)$.*

Now we are ready to prove Lemma 5:

Proof of Lemma 5. Let us simulate Algorithm 1. Let $E_i = \{1, 2, \ldots, i\}$ be the set of the first i elements observed by the algorithm, and similar to the proof of Theorem 1, let S_{i-1} be the independent set chosen by the algorithm after observing the elements of E_{i-1}. Recall that S_{i-1} is a base of E_{i-1}.

Since $S_{i-1} \subset E_{i-1}$, by Corollary 9, $span(S_{i-1}) \subseteq span(E_{i-1})$. Therefore if $i \notin span(E_{i-1})$, then $i \notin span(S_{i-1})$, and thus by Lemma 7, $S_{i-1} + i \in \mathcal{I}$ is an independent set. Therefore i will be chosen by the algorithm.

Conversely, suppose $i \in span(E_{i-1})$. Since S_{i-1} is a base of E_{i-1}, by Lemma 6 it is also a base of E_i. Hence $S_{i-1} + i \notin \mathcal{I}$ is not independent, and i will not be chosen by the algorithm. $\square$

3 Properties of the rank function

In the section we prove some of the basic properties of rank functions.

Lemma 10 *The rank function of a matroid satisfies the following:*

1. *For any $S \subseteq T \subseteq E$ of elements, we have $r(S) \leq r(T)$ (**monotonicity**)*
2. *For any $S, T \subseteq E$, we have $r(S \cup T) + r(S \cap T) \leq r(S) + r(T)$ (**submodularity**)*

Proof: The monotonicity property is trivial (since any base of S is also an independent set of T). To prove the submodularity property we use the previous lemma. By Lemma 8 we know that if $r(S+i) = r(S)$ for some elements $i \in E$, then $r(T+i) = r(T)$. Since for any set S, and any element $i \in E$, we have $r(S+i) - r(S) \leq 1$, this implies that for any $S \subset T$ and $i \in E \setminus T$ we have:

$$r(T+i) - r(T) \leq r(S+i) - r(S).$$

In the next lemma we show that this is equivalent to submodularity. $\square$

Lemma 11 *Let $f : 2^E \to \mathbb{R}$ be a set function on a ground set E. Then f is submodular ($\forall A, B \subseteq E,\ f(A \cap B) + f(A \cup B) \leq f(A) + f(B)$) if and only if for all $S \subset T \subset E$ and $i \in E \setminus T$:*

$$f(T + i) - f(T) \leq f(S + i) - f(S).$$

Proof: Assume for all $S \subset T$ and $i \notin T$, we have $f(T+i) - f(T) \leq f(S+i) - f(S)$. Let $A, B \subseteq E$ be two subsets of E. If $B \subseteq A$, the claim is trivial. Let $B \setminus A = \{b_1, b_2, \ldots, b_k\}$. We have:

$$\begin{aligned} f(A \cup B) - f(A) &= \sum_{i=1}^{k} (f(A + b_1 + \ldots + b_i) - f(A + b_1 + \ldots + b_{i-1})) \\ &\leq \sum_{i=1}^{k} (f(A \cap B + b_1 + \ldots + b_i) - f(A \cap B + b_1 + \ldots + b_{i-1})) \\ &= f(B) - f(A \cap B). \end{aligned}$$

Here the inequality follows from the assumption once we set $S := A \cap B$ and $T := A$ (note that this implies $S \subseteq T$).

Conversely, suppose for any two sets A, B we have $f(A \cup B) + f(A \cap B) \leq f(A) + f(B)$. Let $S \subset T$ and $i \notin T$. Now set $A := S + i$ and $B := T$. By the submodularity condition, we get:

$$f(T + i) + f(S) = f(A \cup B) + f(A \cap B) \leq f(A) + f(B) = f(S + i) + f(T),$$

which completes the proof. $\square$