**CS 369P: Polyhedral techniques in combinatorial optimization**
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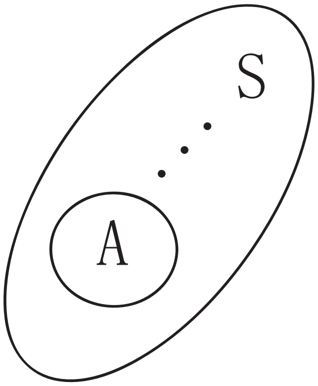
**Lecture date: Oct 19, 2010** **Scribe: Hao Su**

## 1 Characterization of rank functions

**Lemma 1** *A function $r : 2^E \mapsto \mathbb{R}$ is a rank function of a matroid if and only if*

1. $r(\emptyset) = 0$ *and* $r(S + i) - r(S) \in \{0, 1\}$ *for all* $S, i$
2. $r$ *is submodular, i.e.* $r(S \cup T) + r(S \cap T) \leq r(S) + r(T)$ *for all* $S, T$.



**Proof:** We already know from the previous lecture that any rank function must satisfy these conditions. So let us assume that $r$ satisfies the conditions and define $\mathcal{I} = \{A : r(A) = |A|\}$. We claim that $(E, \mathcal{I})$ is a matroid and $r$ is its rank function.

By the first condition on $r$, is it clear that $\mathcal{I}$ is closed under taking subsets. We claim that for any set $S$, all maximal subsets of $S$ which are in $\mathcal{I}$ (bases of $S$) have the same size. Consider any $A \subset S$ such that $r(A) = |A|$. We have $r(A + i) - r(A) \in \{0, 1\}$. As long as $r(A) < r(S)$, by submodularity there is $i \in S \setminus A$ such that $r(A + i) = r(A) + 1$. Hence, if $A$ is maximal such that $r(A) = |A|$, we have $r(A) = r(S)$. i.e., all bases have the same size, and $r(S)$ is equal to that size. $\square$

## 2 The dual matroid

**Definition 2** *For a matroid,* $\mathcal{M} = (E, \mathcal{I})$*, the dual matroid* $\mathcal{M}^* = (E, \mathcal{I}^*)$ *is a matroid such that the bases of* $\mathcal{M}^*$ *are exactly the complements of the bases of* $\mathcal{M}$.

**Theorem 3** $\mathcal{M}^*$ *is a matroid and its rank function is*

$$r^*(S) = |S| - (r(E) - r(E \setminus S)).$$

**Proof:** First, we show that $r^*$ is the rank function of a matroid. Its marginal values are in $\{0, 1\}$, and it's submodular, because

$$\begin{aligned} r^*(S \cup T) + r^*(S \cap T) &= |S \cup T| + |S \cap T| - 2r(E) + r(E \setminus (S \cup T)) + r(E \setminus (S \cap T)) \\ &= |S| + |T| - 2r(E) + r((E \setminus S) \cap (E \setminus T)) + r((E \setminus S) \cup (E \setminus T)) \\ &\leq |S| + |T| - 2r(E) + r(E \setminus S) + r(E \setminus T) \\ &= r^*(S) + r^*(T). \end{aligned}$$

This implies that $\mathcal{I}^* = \{I : r^*(I) = |I|\}$ are the independent sets of some matroid, in fact exactly the dual matroid, since

$$\begin{aligned} I \text{ is a base of } \mathcal{M}^* &\Leftrightarrow r^*(I) = |I| = r^*(E) \\ &\Leftrightarrow r(E \setminus I) = r(E) \text{ and } |I| = |E| - r(E) \\ &\Leftrightarrow E \setminus I \text{ is a base of } \mathcal{M}. \end{aligned}$$

□

**Definition 4** *For $e \in E$,*

1. *the "deletion" of $e$ produces*

$$\mathcal{M} \setminus e = (E - e, \{I \subseteq E - e : I \in \mathcal{I}\})$$

2. *the "contraction" of $e$ produces*

$$\mathcal{M}/e = (E - e, \{I \subseteq E - e : I + e \in \mathcal{I}\})$$

*if $r(\{e\}) = 1$, and $\mathcal{M}/e = \mathcal{M} \setminus e$ if $r(\{e\}) = 0$.*

**Lemma 5** *Contraction is dual to deletion, i.e.*

$$(M/e)^* = M^* \setminus e$$

**Proof:** Assume $r(\{e\}) = 1$, otherwise deleting/contracting $e$ does not affect independence on either side. We have:
$I$ is independent in $(\mathcal{M}/e)^*$
$\Leftrightarrow I \subseteq B$ for some base $B$ of $(\mathcal{M}/e)^*$
$\Leftrightarrow I \cap B' = \emptyset$ for some base $B'$ of $\mathcal{M}/e$ and $e \notin I$
$\Leftrightarrow I \cap B'' = \emptyset$ for some base of $\mathcal{M}$, $B'' = B' + e$
$\Leftrightarrow I \subseteq B'''$ for some base $B'''$ of $\mathcal{M}^*$ and $e \notin I$. □

How does deletion/contraction affect the rank function?
$r_{\mathcal{M} \setminus e}(S) = r_{\mathcal{M}}(S)$, for any $S \subseteq E - e$.
$r_{\mathcal{M}/e}(S) = r_{\mathcal{M}}(S + e) - r_{\mathcal{M}}(e)$, for any $S \subseteq E - e$.

Matroid duality in a nutshell

| | | |
|---|---|---|
| *contraction* | $\leftrightsquigarrow$ | *deletion* |
| *base* | $\leftrightsquigarrow$ | *complement of a base* |
| *independent set* | $\leftrightsquigarrow$ | *complement of a spanning set* |
| *circuit (minimal dependent set)* | $\leftrightsquigarrow$ | *cut (minimal, intersects every base)* |
| *loop (no independent set contains it)* | $\leftrightsquigarrow$ | *bridge (every base contains it)* |

(The terminology comes from graphic matroids.)

## 3 The matroid polytope

**Theorem 6** *For a matroid* $\mathcal{M} = (E, \mathcal{I})$,

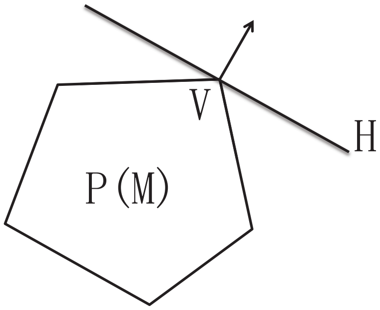
$$P(\mathcal{M}) = \{\mathbf{x} \in \mathbb{R}_+^E : \forall S \subseteq E, x(S) \leq r_{\mathcal{M}}(S)\}$$

*where* $P(M) \equiv conv\{\mathbf{x}_I : I \in \mathcal{I}\}$ *and* $r_{\mathcal{M}}$ *is the rank function of* $\mathcal{M}$.

**Proof:** We show that for any weight function $\mathbf{w} \in \mathbb{R}^E$

$$\begin{array}{rcl} max & & \mathbf{w}^T\mathbf{x} \\ \forall S; x(S) & \leq & r(S), \\ \mathbf{x} & \geq & 0 \end{array}$$

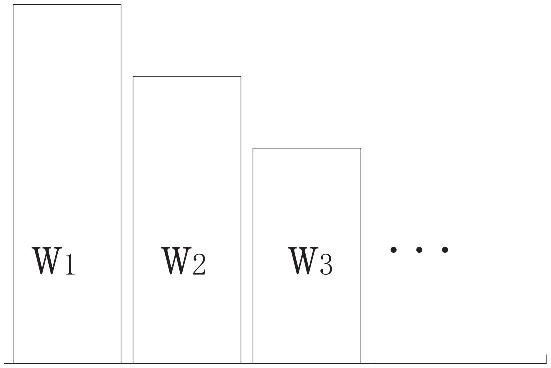
has an optimal solution in $\{0,1\}^E$.



For any vertex $v$, there is some direction $\mathbf{w}$ such that $v$ is the unique optimum of $max\{\mathbf{w}^T\mathbf{x} : \mathbf{x} \in P\}$, Hence, this will imply that all vertices are in $\{0,1\}^E$

Consider LP duality: (for now, let $\mathbf{w} \geq 0$)

$$\begin{array}{cc} max\ \mathbf{w}^T\mathbf{x} : & min\ \sum_S y_S r(S) : \\ \forall S; \sum_{i \in S} x_i \leq r(S), & \forall i; \sum_{S: i \in S} y_S \geq w_i, \\ \mathbf{x} \geq 0 & \mathbf{y} \geq 0 \end{array}$$



Define a primal solution by the greedy algorithm:

$$w_1 \geq w_2 \geq w_3 \geq ...$$

$$x_i = \begin{cases} 1 & \text{if greedy takes } i \\ 0 & \text{otherwise} \end{cases}$$

($\mathbf{x} \in P(\mathcal{M})$ clearly).

$$\begin{aligned} \text{Greedy takes element i} &\Leftrightarrow i \notin span(\{1,2,...,i-1\}) \\ &\Leftrightarrow r(\{1,2,...,i\}) = r(\{1,2,...,i-1\}) + 1 \end{aligned}$$

So:

$$\begin{aligned} \mathbf{w}^T\mathbf{x} &= \sum_{i=1}^{n} w_i(r([i]) - r([i-1])) \\ &= \sum_{j=1}^{n} r([j])(w_j - w_{j+1}) \end{aligned}$$

This suggests a dual solution: $y_{[j]} = w_j - w_{j+1}, \forall j$ (formally, $w_{n+1} = 0$)
We have:

$$\sum_{S:i\in S} y_S = \sum_{j=1}^{n} y_{[j]} = \sum_{j=1}^{n}(w_j - w_{j+1}) = w_i,$$

$$\sum_{S} y_S r(S) = \sum_{j=1}^{n} y_{[j]} r([j]) = \mathbf{w}^T\mathbf{x}.$$

This indicates both $\mathbf{x}, \mathbf{y}$ are optimal primal/dual solutions.

For $\mathbf{w} \in \mathbb{R}^E$, note that the primal optimum does not change if we replace $w_i < 0$ by $w_i = 0$; in any case, $x_i = 0$. Therefore, there is an optimal solution in $\{0,1\}^E$ for any $\mathbf{w} \in \mathbb{R}^E$. $\square$

**Note:** We have proved more: if $\mathbf{w} \in \mathbb{R}^E$, then both primal and dual optima are integers. ("a totally dual integral system")

**Corollary 7** *The matroid base polytope $P_{base}(\mathcal{M})$ is*

$$P_{base}(\mathcal{M}) = \{\mathbf{x} \in \mathbb{R}_+^E : \forall S; x(S) \leq r(S), x(E) = r(E)\}.$$

**Proof:** Obviously, $\chi_B \in P_{base}(\mathcal{M})$ for every base $B$. Suppose that $\mathbf{x}$ satisfies the constraints $\forall S; x(S) \leq r(S)$, and $x(E) = r(E)$. In particular, $\mathbf{x} \in P(\mathcal{M})$, the matroid polytope, and

$$\mathbf{x} = \sum_{I \in \mathcal{I}} \alpha_I \mathbf{x}_I, \sum \alpha_I = 1, \alpha_I \geq 0.$$

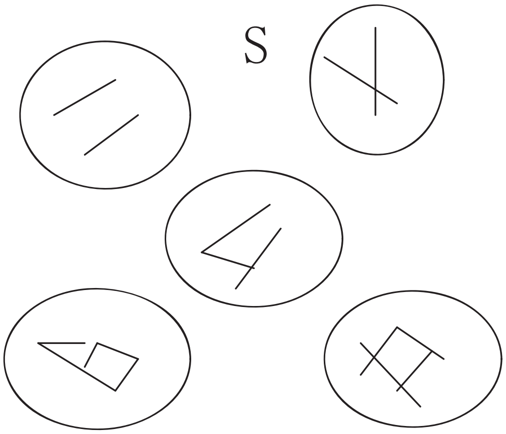
We have $x(E) = \sum_{I \in \mathcal{I}} \alpha_I |I| \leq \sum_{I \in \mathcal{I}} \alpha_I r(E) = r(E)$. But $x(E) = r(E)$, so we have an equality for each $I$ with a positive coefficient $\alpha_I$ and each such $I$ must be a base. $\square$

What does it mean for spanning trees?

$$M = graphic\ matroid, G = (V, E)$$

$$r_{\mathcal{M}}(S) = |V| - \#components\ in\ (V, S)$$

$$r_{\mathcal{M}}(S) = |V| - 1$$



If $S = E[W]$, then $r_{\mathcal{M}}(S) = |W| - 1$.

$$x(S) \leq r_{\mathcal{M}}(S) = |V| - \#components$$

can be obtained by adding up $x(E[W_i]) \leq |W_i| - 1$ over all components of $W_i$.
$\rightsquigarrow$ it is sufficient to keep $x(E[W_i]) \leq |W_i| - 1$.

**Corollary 8** *The forest polytope of $G = (V, E)$ is given by*

$$P_{forest}(G) = \{\mathbf{x} \in \mathbb{R}_+^E : \forall W \subseteq V; x(E[W]) \leq |W| - 1\}$$

*The spanning tree polytope is given by:*

$$P_{spanningtree}(G) = \{\mathbf{x} \in \mathbb{R}_+^E : \forall W \subseteq V; x(E[W]) \leq |W| - 1, x(E) = |V| - 1\}$$

**Note:** All these descriptions are exponentially large, but the polytopes are "nice". We can optimize over them, and by polarity we can also solve the separation problem.