MATH 108: Introduction to Combinatorics, Winter 2016
HOMEWORK 1
Due Friday, January 15

You should solve the homework on your own. Please don't use the internet or any books except Knuth.

Problem 1. [from Monday's class] Consider the ways to partition the set $\{1, 2, \dots, n\}$ recursively into pairs of nonempty sets, until we end up with singletons. (For example, $\{1, 2, 3\}$ can be partitioned into two nonempty sets A, B, $|A| = 1$, $|B| = 2$, and then B can be partitioned into two singletons.) Prove that the number of such partitioning schemes is $(2n-3)!! = (2n-3)(2n-5)(2n-7)\cdots 3 \cdot 1$.

Problem 2. [Knuth page 36, problem 1.] Suppose $n = 4m - 1$. Construct arrangements of Langford pairs for the numbers $\{1, 1, \dots, n, n\}$ with the property that we also obtain a solution for $n = 4m$ by changing the first '$2m - 1$' to '$4m$' and appending $2m - 1, 4m$ at the right.
Hint: start with $4m-4, 4m-6, \dots, 2m$ (assuming $m > 1$). The first four solutions (in hexadecimal notation) are: 231213, 46171435623725, 86a31b1368597a425b2479, ca8e531f1358ac7db9e6427f2469bd.

Problem 3. [Knuth page 36, problem 10.] A *magic square* is an $n \times n$ array containing the integers $\{1, 2, \dots, n^2\}$ such that the sum of each row, each column and each diagonal is the same. Construct a 4×4 magic square, using a 4×4 Greco-Latin square as a guide:

$$\begin{array}{cccc} J\diamondsuit & A\heartsuit & K\spadesuit & Q\clubsuit \\ Q\spadesuit & K\clubsuit & A\diamondsuit & J\heartsuit \\ A\clubsuit & J\spadesuit & Q\heartsuit & K\diamondsuit \\ K\heartsuit & Q\diamondsuit & J\clubsuit & A\spadesuit \end{array}$$

More generally, prove that a "weak" magic square (without the diagonal conditions) exists for any $n \neq 2, 6$, using the existence of Greco-Latin squares.
Hint: Construct a number in $\{1, \dots, n^2\}$ from two numbers in $\{1, \dots, n\}$.

Problem 4. [Knuth page 44, problem 133.] Consider the *musical graph*: $V = \{a_0, \dots, a_{11}, b_0, \dots, b_{11}\}$, with edges (a_i, b_i), $(a_i, a_{i+1 \bmod 12})$, $(b_i, b_{i+1 \bmod 12})$, $(a_i, b_{i+1 \bmod 12})$, $(b_i, a_{i+1 \bmod 12})$ for $0 \le i < 12$. Prove that its chromatic number is 4 and the chromatic index (edge-coloring number) is 5. (Indicate a proper coloring by picture, and prove that there is no coloring using fewer colors.)