MATH 108: Introduction to Combinatorics, Winter 2016
HOMEWORK 7
Due Monday, March 7

You should solve the homework on your own. Don't use any books or the internet.

Problem 1. [Knuth p. 408, variant of problem 11]
Let c_n denote the number of ways that an amount of n cents can be paid in coins of value $5, 10$ and 25 cents.

- Find a simple form for the generating function $C(x) = \sum_{n=0}^{\infty} c_n x^n$.
- Expand the power series explicitly up to x^{25}; how many ways are there to pay 25 cents?

Problem 2. Let $a_k(n)$ denote the number of partitions of an integer n into *at most* k parts. (Let's define $a_k(0) = 1$.)

- Prove that $a_k(n)$ is equal to the number of partitions of n into a sum of integers in $\{1, 2, \ldots, k\}$.
- Find a simple form for the generating function $A_k(x) = \sum_{n=0}^{\infty} a_k(n) x^n$.

Problem 3. [Knuth p. 408, problem 9]
If $(a_1, \ldots, a_m)$ and $(b_1, \ldots, b_m) = (a_1, \ldots, a_m)^T$ are conjugate partitions (i.e., m is the number of rows and columns in both, and $a_1 \geq a_2 \geq \ldots a_m$, $b_1 \geq b_2 \geq \ldots b_m$ are the row/column lengths), prove that the multisets $\{a_1 + 1, \ldots, a_m + m\}$, $\{b_1 + 1, \ldots, b_m + m\}$ are equal (i.e., they contain each integer the same number of times).

Problem 4. Call two permutations $\sigma_1, \sigma_2 \in \mathbb{S}_n$ *conjugate* ($\sigma_1 \sim \sigma_2$) if there is a permutation π such that $\sigma_2 = \pi^{-1} \sigma_1 \pi$.

- Show that $\sigma_1 \sim \sigma_2$ is an equivalence relation (symmetric, reflexive and transitive).
- Show that $\sigma_1 \sim \sigma_2$ if and only if σ_1 and σ_2 have the same cycle structure (lengths of cycles, written down as a monotone sequence).
- Show that the number of equivalence classes of $\sim$ ("conjugacy classes" in $\mathbb{S}_n$) is exactly $p(n)$, the number of partitions of n into positive integers.