MATH 113: Linear Algebra, Autumn 2018
Midterm exam - sample questions

Please try to do all 8 problems. Show your work (partial credit will be given). All problems count equally. No notes or books allowed except one handwritten 3 × 5" card of notes (both sides). Please hand it in with your exam. Good luck!

Problem 1. Assume that $S, T \in \mathcal{L}(V)$ are operators such that $\operatorname{range}(S) \subseteq \operatorname{null}(T)$. Prove that $(ST)^2 = 0$.

Solution: We need to show $STST = 0$.
By definition $\operatorname{range}(S)$ is $\{Sv : v \in V\}$ and $\operatorname{null}(T)$ is $\{v \in V : Tv = 0\}$. Hence $\operatorname{range}(S) \subseteq \operatorname{null}(T)$ means that $T(Sv) = 0$ for every $v \in V$. From this, we conclude that for every $v \in V$,

$$(STST)v = S(T(S(Tv)) = S(0) = 0.$$

Problem 2. Assume V is finite-dimensional and $S, T, U \in \mathcal{L}(V)$. Prove that $STU = I$ then T is invertible and $T^{-1} = US$.

Solution: First, we show that U is injective. Indeed, if $Uv = 0$, then

$$v = Iv = (STU)v = (ST)(Uv) = (ST)(0) = 0.$$

Since any injective operator on a finite-dimensional space is invertible, we conclude that U is invertible.
Next, we show that S is surjective. Indeed, if $v \in V$, then

$$v = Iv = (STU)v = S((TU)(v)).$$

Since any surjective operator on a finite-dimensional space is invertible, we conclude that S is invertible.
Hence, we obtain

$$T = ITI = (S^{-1}S)T(UU^{-1}) = S^{-1}(STU)U^{-1} = S^{-1}IU^{-1} = S^{-1}U^{-1}.$$

Finally, using the rules for the inverse of a product, we obtain

$$T^{-1} = (S^{-1}U^{-1})^{-1} = (U^{-1})^{-1}(S^{-1})^{-1} = US.$$

Problem 3. Assume V, W finite-dimensional, $\phi \in W'$ and $T \in \mathcal{L}(V, W)$ such that $\operatorname{null}(T') = \operatorname{span}(\phi)$. Prove that $\operatorname{range}(T) = \operatorname{null}(\phi)$.

Solution: First, we show that $\operatorname{range}(T) \subseteq \operatorname{null}(\phi)$ – i.e., that $\phi(Tv) = 0$ for every $v \in V$.
Since $\phi \in \operatorname{span}(\phi)$ and $\operatorname{span}(\phi) = \operatorname{null}(T')$, we have $\phi \in \operatorname{null}(T')$ – that is, $T'\phi = 0$. By definition $(T'\phi)(v) = \phi(Tv)$. Hence, we have that $\phi(Tv) = 0$ for every $v \in V$, as desired.
Since (T) is finite-dimensional and $\operatorname{range}(T) \subseteq \operatorname{null}(T)$, in order to prove that $\operatorname{range}(T) = \operatorname{null}(T)$, it now suffices to show that $\dim \operatorname{range}(T) = \dim \operatorname{null}(T)$. We do this using several dimension formulae, as well as our assumption $\operatorname{null}(T') = \operatorname{span}(\phi)$:

$$\begin{aligned}\dim \operatorname{range}(T) &= \dim \operatorname{range}(T') = \dim(W') - \dim \operatorname{null}(T') = \dim(W) - \dim \operatorname{null}(T') \\ &= \dim(W) - \dim \operatorname{span}(\phi) = \dim(W) - \dim \operatorname{range}(\phi) = \dim \operatorname{null}(\phi)\end{aligned}$$

The one formula that we used which isn't a theorem from class is $\dim \operatorname{span}(\phi) = \dim \operatorname{range}(\phi)$. But this is clear, since $\operatorname{span}(\phi)$ and $\operatorname{range}(\phi)$ are both 0-dimensional in case $\phi = 0$ and 1-dimensional in case $\phi \neq 0$.

Problem 4. Prove that for any operator $T \in \ \in \mathcal{L}(V)$, if $T' \in \mathcal{L}(V')$ is the dual map to T, then $\det T = \det T'$.

Solution: Let A be the matrix of T with respect to any basis. Regardless of whether the ground field is the real or the complex numbers, we can treat A as a matrix with complex entries. Hence, we know that A can be transformed into an upper-triangular matrix: $RAR^{-1} = D$, where D is upper-triangular and R is invertible (both possibly having complex entries). Since determinants are multiplicative, we then have

$$\det(T) = \det(A) = \det(R)\det(A)\det(R)^{-1} = \det(RAR^{-1}) = \det(D)$$

Now the matrix of T' with respect to the dual basis of V' to our chosen basis of V is just A^t. Using the compatibility of inversion with transposition, and the rule for the transpose of a product, we have

$$D^t = (RAR^{-1})^t = (R^{-1})^t A^t R^t = (R^t)^{-1} A^t R^t$$

Using again that determinants are multiplicative, we then have

$$\det(T') = \det(A^t) = \det(R^t)^{-1}\det(A^t)\det(R^t) = \det((R^t)^{-1}A^tR^t) = \det(D^t)$$

But now, since D and D^t are upper-triangular and lower-triangular, respectively, their determinant is given by the product of the diagonal entries, which are the same. Hence, $\det(D) = \det(D^t)$, and hence $\det(T) = \det(T')$.

Problem 5. Prove that if $T \in \mathcal{L}(V)$ is diagonalizable then $V = \text{null}(T) \oplus \text{range}(T)$. How can you describe null(T) and range(T) in terms of the eigenvectors of T?

Solution: Since T is diagonalizable, there is by definition a basis $v_1, \dots, v_n$ of eigenvectors for T, so $Tv_i = \lambda_i$ for some scalars $\lambda_1, \dots, \lambda_n$. By rearranging the v_i, we can assume that $\lambda_1 = \cdots = \lambda_k = 0$ and $\lambda_{k+1}, \dots, \lambda_n \neq 0$ for some k.

Clearly, $V = \text{span}(v_1, \dots, v_k) \oplus \text{span}(v_{k+1}, \dots, v_n)$, so it only remains to show that $\text{null}(T) = \text{span}(v_1, \dots, v_k)$ and $\text{range}(T) = \text{span}(v_{k+1}, \dots, v_n)$. This will also give us the desired description of null(T) and range(T) in terms of eigenvectors.

We have $\text{span}(v_1, \dots, v_k) \subseteq \text{null}(T)$, since for any scalars $c_1, \dots, c_k$,

$$T(c_1v_1 + \cdots + c_kv_k) = c_1Tv_1 + \cdots + c_kTv_k = 0$$

We have $\text{span}(v_{k+1}, \cdots, v_n) \subseteq \text{range}(T)$, since for any scalars $c_{k+1}, \dots, c_n$,

$$c_{k+1}v_{k+1} + \cdots + c_nv_n = \frac{c_{k+1}}{\lambda_{k+1}}(\lambda_{k+1}v_{k+1}) + \cdots + \frac{c_n}{\lambda_n}(\lambda_nv_n) = T\left(\frac{c_{k+1}}{\lambda_{k+1}}v_{k+1} + \cdots + \frac{c_n}{\lambda_n}v_n\right)$$

(here, we are using that $\lambda_{k+1}, \dots, \lambda_n \neq 0$).

It now suffices to show that $\dim \text{null}(T) = k$ and $\dim \text{range}(T) = n - k$. From what we have shown, we have $\dim \text{null}(T) \geq k$ and $\dim \text{range}(T) \geq n - k$. But these inequalities must both be equalities since $\dim \text{range}(T) + \dim \text{null}(T) = \dim(V) = n$.

Problem 6. Let $T \in \mathcal{L}(\mathbb{R}^2)$ be such that $T(x, y) = (y, x + y)$. Find the eigenvectors and eigenvalues of T.

Solution: Suppose (x, y) is an eigenvector for T with eigenvalue λ. We then have

$$\begin{aligned} &\text{(I)} \quad \lambda x = y \\ &\text{(II)} \quad \lambda y = x + y. \end{aligned}$$

Substituting (I) into (II), we obtain

$$x\lambda^2 - x\lambda - x = 0.$$

Note that we cannot have $x = 0$, since then by (I) we would have $y = 0$, but $(0, 0)$ is by definition never an eigenvector. Hence, by the quadratic formula,

$$\lambda = \frac{x \pm \sqrt{x^2 + 4x^2}}{2x} = \frac{1 \pm \sqrt{5}}{2}$$

Hence, the only possible eigenvalues are $\lambda_1 = \frac{1+\sqrt{5}}{2}$ and $\lambda_2 = \frac{1-\sqrt{5}}{2}$. By (I), the eigenvectors of λ_i must be of the form $(t, \lambda_i t)$, with t a non-zero scalar. And indeed, these are all eigenvectors:

$$T(t, \lambda_i t) = (\lambda_i t, (1 + \lambda_i)t) = (\lambda_i t, \lambda_i^2 t) = \lambda_i(t, \lambda_i t)$$

where we have used $\lambda_i^2 = 1 + \lambda_i$.

Problem 7. Suppose $\|u\| = \|v\|$. Prove that $\|\alpha u + \beta v\| = \|\beta u + \alpha v\|$ for every $\alpha, \beta \in \mathbb{R}$. Is the same true for every $\alpha, \beta \in \mathbb{C}$?

Solution: Since $\|x\| = \sqrt{\|x\|^2}$ for every x, it suffices to show that $\|\alpha u + \beta v\|^2 = \|\beta u + \alpha v\|^2$. Now (whether or not α, β are real) we have

$$\|\alpha u + \beta v\|^2 = \langle \alpha u + \beta v, \alpha u + \beta v \rangle = |\alpha|^2 \langle u, u \rangle + |\beta|^2 \langle v, v \rangle + \alpha\bar{\beta} \langle u, v \rangle + \beta\bar{\alpha} \langle v, u \rangle$$

and similarly

$$\|\beta u + \alpha v\|^2 = |\beta|^2 \langle u, u \rangle + |\alpha|^2 \langle v, v \rangle + \beta\bar{\alpha} \langle u, v \rangle + \alpha\bar{\beta} \langle v, u \rangle .$$

Since by assumption $\langle u, u \rangle = \langle v, v \rangle$, these are equal as long as $\alpha = \bar{\alpha}$ and $\beta = \bar{\beta}$ – that is, if α, β are real.

However, if α, β are not real, they need not be equal. For example, taking our vector space to be $\mathbb{C}$ with the standard inner product $\langle w, z \rangle = w\bar{z}$, and taking $\alpha = v = i$ and $\beta = u = 1$, we have

$$\|\alpha u + \beta v\| = \|i + i\| = 2$$

but

$$\|\beta u + \alpha v\| = \|1^2 + i^2\| = \|0\| = 0.$$

Problem 8. Suppose that T is a normal operator with eigenvalues 3 and 4. Prove that there exists $v \in V$ such that $\|v\| = \sqrt{2}$ and $\|Tv\| = 5$.

Solution: The one property that we use of normal operators is that their eigenvectors with distinct eigenvalues are orthogonal to one another.

In particular, taking eigenvectors u, w of T with eigenvalues 3 and 4 and with $\|u\| = \|w\| = 1$, and setting $v = u + w$, we have

$$\|v\|^2 = \|u + w\|^2 = \|u\|^2 + \|w\|^2 = 2$$

where the second equality is by the Pythagorean theorem, since u and v are orthogonal, and

$$\|Tv\|^2 = \|Tu + Tw\|^2 = \|3u + 4w\|^2 = \|3u\|^2 + \|4w\|^2 = 9 + 16 = 25.$$

where we again used the Pythagorean theorem.