Math 113 Autumn 2018
FINAL EXAM INFORMATION

For the final exam, you are supposed to know all the topics we have covered in class. Any result *stated in class* can be used on the exam without re-proving it. (This does not include homework problems or other results stated in the book.) The list of topics is as follows:

- Fields, vector spaces and subspaces (1.A, 1.B, 1.C)
- Span, linear independence, basis and dimension (2.A, 2.B, 2.C)
- Linear maps, matrices, null space, range, fundamental theorem (3.A, 3.B, 3.C)
- Invertible linear maps, isomorphic vector spaces, linear operators (3.D)
- Products, quotients, duality, annihilators, matrix rank (3.E, 3.F)
- Facts about polynomials — without proofs (4.)
- Eigenvalues and eigenvectors, upper-triangular matrices (5.A, 5.B)
- Determinants, independence under change of basis (10.B)
- Inner product, norm, orthogonality, Cauchy-Schwartz inequality (6.A)
- Orthonormal basis, the Gram-Schmidt procedure, Riesz' representation theorem (6.B)
- Self-adjoint and normal operators, the spectral theorem (complex and real) (7.A, 7.B)

On the following page you can find a sample exam for practice.

MATH 113: Linear Algebra, Autumn 2018
Midterm exam - sample questions

Please try to do all 8 problems. Show your work (partial credit will be given). All problems count equally. No notes or books allowed except one handwritten 3×5" *card of notes (both sides). Please hand it in with your exam. Good luck!*

Problem 1. Assume that $S, T \in \mathcal{L}(V)$ are operators such that $\text{range}(S) \subseteq \text{null}(T)$. Prove that $(ST)^2 = 0$.

Problem 2. Assume V is finite-dimensional and $S, T, U \in \mathcal{L}(V)$. Prove that if $STU = I$ then T is invertible and $T^{-1} = US$.

Problem 3. Assume V, W finite-dimensional, $\phi \in V'$ and $T \in \mathcal{L}(V, W)$ such that $\text{null}(T') = \text{span}(\phi)$. Prove that $\text{range}(T) = \text{null}(\phi)$.

Problem 4. Prove that for any operator $T \in \mathcal{L}(V)$, if $T' \in \mathcal{L}(V')$ is the dual map to T, then $\det T' = \det T$.

Problem 5. Prove that if T is diagonalizable then $V = \text{null}(T) \oplus \text{range}(T)$. How can you describe $\text{null}(T)$ and $\text{range}(T)$ in terms of the eigenvectors of T?

Problem 6. Let $T \in \mathcal{L}(\mathbb{R}^2)$ be such that $T(x, y) = (y, x + y)$. Find the eigenvectors and eigenvalues of T.

Problem 7. Suppose $\|u\| = \|v\|$. Prove that $\|\alpha u + \beta v\| = \|\beta u + \alpha v\|$ for every $\alpha, \beta \in \mathbb{R}$. Is the same true for every $\alpha, \beta \in \mathbb{C}$?

Problem 8. Suppose that T is a normal operator with eigenvalues 3 and 4. Prove that there exists $v \in V$ such that $\|v\| = \sqrt{2}$ and $\|Tv\| = 5$.