MATH 113: Linear Algebra, Autumn 2018
OPTIONAL HOMEWORK 8 — only for practice, will not be graded

Problem 1. Suppose $u, v \in V$ for an inner product space over $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$. Prove that u is orthogonal to v if and only if

$$\|u\| = \min\{\|u + \alpha v\| : \alpha \in \mathbb{F}\}.$$

Problem 2. Suppose that $\|u\| = 3$, $\|u + v\| = 4$ and $\|u - v\| = 6$. Compute $\|v\|$.

Problem 3. On $\mathcal{P}_2(\mathbb{R})$, consider the inner product $\langle p, q\rangle = \int_0^1 p(x)q(x)dx$. Find an orthonormal basis such that the differentiation operator has an upper-triangular matrix.

Problem 4. Suppose $T \in \mathcal{L}(V)$, V finite-dimensional over $\mathbb{C}$. Prove that $\lambda \in \mathbb{C}$ is an eigenvalue of T (on a complex inner product space) if and only if $\overline{\lambda}$ is an eigenvalue of T^*.

Problem 5. Suppose S and T are self-adjoint operators. Prove that ST is self-adjoint if and only if $ST = TS$.

Bonus problem. Suppose that V is an inner product space over $\mathbb{R}$ and $v_1, \ldots, v_n$ are linearly independent in V. Prove that there exists $w \in V$ such that $\langle w, v_j\rangle > 0$ for all $1 \leq j \leq n$.