# MATH 113: Linear Algebra, Autumn 2018
## Midterm exam - Monday October 29, 11:30 - 12:25

**Problem 1.** Find a basis of the following subspace of $\mathbb{C}^4$:

$$U = \{x \in \mathbb{C}^4 : x_1 + 2x_2 = 0, x_3 + ix_4 = 0\}.$$

*Solution:* For any $(x_1, x_2, x_3, x_4) \in U$, we have $x_1 = -2x_2$ and $x_3 = -ix_4$. So every solution is in the form $(-2x_2, x_2, -ix_4, x_4) = x_2(-2, 1, 0, 0) + x_4(0, 0, -i, 1)$. This means that $U$ is spanned by the vectors $(-2, 1, 0, 0), (0, 0, -i, 1)$. Also, these 2 vectors are linearly independent, because $\alpha(-2, 1, 0, 0) + \beta(0, 0, -i, 1) = 0$ clearly implies $\alpha = \beta = 0$. So $(-2, 1, 0, 0), (0, 0, -i, 1)$ is a basis of $U$.

**Problem 2.** Suppose that $p_0(x), \ldots, p_m(x)$ are polynomials in $\mathcal{P}_m(\mathbb{R})$ such that $p_j(1) = 0$ for all $0 \leq j \leq m$. Prove that $p_0(x), \ldots, p_m(x)$ are not linearly independent.

*Solution:* Let $U = \{p \in \mathcal{P}_m(\mathbb{R}) : p(1) = 0\}$. $U$ is a subspace of $\mathcal{P}_m(\mathbb{R})$, because taking sums and scalar multiplies preserves the condition $p(1) = 0$. Also, $U$ is not equal to the full space $\mathcal{P}_m(\mathbb{R})$, because not every polynomial satisfies $p(1) = 0$. So the dimension of $U$ is strictly less than the dimension of $\mathcal{P}_m(\mathbb{R})$, which is $m + 1$. Hence $\dim(U) \leq m$ and we cannot have $m + 1$ linearly independent vectors in $U$.

**Problem 3.** Prove that if $x, y \in u + U$ for some vectors $x, y, u \in V$ and a subspace $U$ of $V$, then $\alpha x + (1 - \alpha)y \in u + U$ for every $\alpha \in \mathbb{F}$.

*Solution:* If $x, y \in u + U$, we can write $x = u + w$ and $y = u + z$ for some $w, z \in U$. Then $\alpha x + (1 - \alpha)y = \alpha(u + w) + (1 - \alpha)(u + z) = u + \alpha w + (1 - \alpha)z \in u + U$, because $\alpha w + (1 - \alpha)z \in U$ by $U$ being closed under linear combinations.

**Problem 4.** Suppose $V$ is finite-dimensional and $S, T \in \mathcal{L}(V)$. Prove that if $ST = I$ (the identity), then $TS = I$ as well.

*Solution:* We assume that $ST = I$. This implies that $TSTv = Tv$ for every $v \in V$. Equivalently, we can say $TSw = w$ for every $w \in range(T)$.

Finally, we claim that $range(T) = V$. If $range(T) \neq V$, then $\dim(range(T)) < \dim(V)$, and then $\dim(range(ST)) \leq \dim(range(T)) < \dim(V)$, because $range(ST)$ is the range of $S$ as a linear map applied to $range(T)$ (and dimension cannot increase by applying a linear map). But $ST = I$ so $range(ST) = V$.

So it must be the case that $range(T) = V$ and $TSw = w$ for every $w \in V$.