Stanford University **Spring 2016**
Math 233: Non-constructive methods in combinatorics **Instructor: Jan Vondrák**
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Lecture 3. LLL application: Ramsey numbers

As an application of the asymmetric LLL (Theorem 2.8) we will prove a lower bound for Ramsey numbers $R(3, l)$.

For integers $k, l \geq 2$ the Ramsey number $R(k, l)$ is the minimal positive integer N, such that for every edge-coloring of the complete graph K_N there exists a red clique of size k or a blue clique of size l. By induction on k and l, one can easily prove an upper bound

$$R(k, l) \leq \binom{k + l - 2}{k - 1}.$$

We will now consider the case $k = 3$ and $l \geq 3$. Then the above inequality reads $R(3, l) \leq \frac{1}{2}l(l + 1)$. We would like obtain a lower bound for $R(3, l)$ from the LLL.

In order to do so, we consider a complete graph on n vertices and a random coloring of its edges with the colors red and blue: We color each edge red with probability p and blue with probability $1 - p$; independently for all edges. Here the number of vertices n and the probability p will be specified later (depending on the value of l). Our goal is to obtain with positive probability a coloring without a red triangle and without a blue l-clique, since this would establish the lower bound $R(3, l) > n$.

For each 3-element subset T of the vertex set let $\mathcal{A}_T$ be the event that the three vertices in T form a red triangle. Note that for each such T we have $\mathbb{P}(\mathcal{A}_T) = p^3$ and the number of these events $\mathcal{A}_T$ is $\binom{n}{3}$. Furthermore, for each l-element subset S of the vertex set let $\mathcal{B}_S$ be the event that the l vertices in S form a blue clique. Note that for each such S we have

$$\mathbb{P}(\mathcal{B}_S) = (1 - p)^{\binom{l}{2}}$$

and the number of these events $\mathcal{B}_S$ is $\binom{n}{l}$.

Let us now define a dependency graph for these events. We join two events of the form $\mathcal{A}_T$ or $\mathcal{B}_S$, if the corresponding sets S or T intersect in at least two vertices (i.e. if they share an edge). It is clear that an event $\mathcal{A}_T$ or $\mathcal{B}_S$ is mutually independent of all events with which it does not share an edge. Hence the graph we just defined is indeed a dependency graph.

Let us now bound the degrees in this graph. First consider a vertex $\mathcal{A}_T$. It is connected to at most $3n$ events $\mathcal{A}_{T'}$. (In order to be connected, the triangles T and T' need to share an edge. T has 3 edges and for each of them there are at most n choices of a third vertex to form a triangle T'). Also, trivially $\mathcal{A}_T$ is connected to at most $\binom{n}{l}$ events $\mathcal{B}_S$. (We use a trivial bound here since the actual number of dependencies is not much less.) Now, let us consider a vertex $\mathcal{B}_S$. It is connected to at most $\binom{l}{2}n$ events $\mathcal{A}_T$ (In order to have a connection, the sets S and T need to intersect in at least 2 elements. There are $\binom{l}{2}$ choices of two elements in S and for each of them at most n choices for the third vertex in T). Also, trivially $\mathcal{B}_S$ is connected to at most $\binom{n}{l}$ events $\mathcal{B}_{S'}$.

So in order to apply the LLL we need to find positive real numbers $x, y \in (0, 1)$ with

$$p^3 = \mathbb{P}(\mathcal{A}_T) \leq x(1 - x)^{3n}(1 - y)^{\binom{n}{l}}$$

and

$$(1-p)^{\binom{l}{2}} = \mathbb{P}(\mathcal{B}_S) \le y(1-x)^{\binom{l}{2}n}(1-y)^{\binom{n}{l}}.$$

Here x and y shall be the values x_i in Theorem 2.8. Note that we choose the same value x for all events $\mathcal{A}_T$ (since the situation is symmetric for all choices of T) and the same value y for all events $\mathcal{B}_S$ (since the situation is symmetric for all choices of S).

Let us assume that for some value of n we can find positive real numbers $p, x, y \in (0,1)$ such that both of the above inequalities are fulfilled. Then the LLL yields that with positive probability none of the events $\mathcal{A}_T$ and $\mathcal{B}_S$ happen. This means that there is a coloring of the edges of a complete graph on n vertices such that there is no red triangle and no blue l-clique. Therefore we have $R(3,l) > n$, if we can find $p, x, y \in (0,1)$ as above.

Let us now try to find such $p, x, y \in (0,1)$ for n as large as we can. We guess that $y = \frac{1}{\binom{n}{l}}$ would be a smart choice, because then the term $(1-y)^{\binom{n}{l}}$ is roughly constant (around e^{-1}). Furthermore, we observe that p and x need to fulfill the following inequalities:

$$p^3 \le x(1-x)^{3n}(1-y)^{\binom{n}{l}} \le x$$

and

$$e^{-p\binom{l}{2}} \approx (1-p)^{\binom{l}{2}} \le y(1-x)^{\binom{l}{2}n}(1-y)^{\binom{n}{l}} \le (1-x)^{\binom{l}{2}n} \approx e^{-xn\binom{l}{2}}.$$

Hence we need $p \ge xn \ge p^3 n$ and therefore $p \le \frac{1}{\sqrt{n}}$. Finally, to guess the dependence of n on l we note

$$e^{-p\binom{l}{2}} \approx (1-p)^{\binom{l}{2}} \le y(1-x)^{\binom{l}{2}n}(1-y)^{\binom{n}{l}} \le y = \frac{1}{\binom{n}{l}} \approx e^{-l\log n},$$

hence $pl^2 \ge p\binom{l}{2} \ge l \log n$ and therefore $l \ge \frac{1}{p}\log n \ge \sqrt{n}\log n$.

Motivated by this we assume $l \ge 20\sqrt{n}\log n$ and choose $y = \frac{1}{\binom{n}{l}}$, $x = \frac{1}{9n^{3/2}}$ and $p = \frac{1}{3\sqrt{n}}$ (the constants here are not really important, they are just chosen in such a way that the inequalities work out). Then we have

$$(1-y)^{\binom{n}{l}} = \left(1 - \frac{1}{\binom{n}{l}}\right)^{\binom{n}{l}} \ge e^{-1.01}$$

if n is sufficiently large (since $(1-\frac{1}{m})^m \to e^{-1}$ as $m \to \infty$). Furthermore for sufficiently large n we also have

$$(1-x)^{3n} = \left(1 - \frac{1}{9n^{3/2}}\right)^{3n} \ge 1 - \frac{1}{3\sqrt{n}} \ge e^{-0.01}$$

for sufficiently large n. Thus,

$$p^3 = \frac{1}{27n^{3/2}} \le \frac{1}{9n^{3/2}} \cdot e^{-1.02} \le x(1-x)^{3n}(1-y)^{\binom{n}{l}},$$

which establishes the first desired inequality.

For the second inequality note that for sufficiently small $h > 0$ we have $1 - h \ge e^{-2h}$. So for sufficiently large n we get

$$(1-x)^{\binom{l}{2}n} \ge e^{-2xn\binom{l}{2}} = e^{-\frac{2}{9\sqrt{n}}\binom{l}{2}}.$$

Furthermore, using $l \geq 20\sqrt{n}\log n$,

$$y = \frac{1}{\binom{n}{l}} \geq \frac{1}{n^l} = e^{-l\log n} \geq e^{-l^2\frac{1}{20\sqrt{n}}} \geq e^{-l(l-1)\frac{1}{19\sqrt{n}}} \geq e^{-l(l-1)\frac{1}{18\sqrt{n}}+1.01} = e^{-\frac{1}{9\sqrt{n}}\binom{l}{2}+1.01}.$$

Hence

$$(1-p)^{\binom{l}{2}} \leq e^{-p\binom{l}{2}} = e^{-\frac{1}{3\sqrt{n}}\binom{l}{2}} = e^{-\frac{1}{9\sqrt{n}}\binom{l}{2}+1.01} e^{-\frac{2}{9\sqrt{n}}\binom{l}{2}} e^{-1.01} \leq y(1-x)^{\binom{l}{2}n}(1-y)^{\binom{n}{l}},$$

which verifies the second desired inequality.

Thus, for $l \geq 20\sqrt{n}\log n$ we can find $p, x, y \in (0,1)$ with the two desired inequalities. This implies (by the LLL, as described above) that $R(3,l) > n$ whenever $l \geq 20\sqrt{n}\log n$ and n sufficiently large.

Note that $n \leq \frac{l^2}{(40\log l)^2}$ implies

$$20\sqrt{n}\log n \leq 20\frac{l}{40\log l}\log(l^2) = l.$$

Therefore we have $R(3,l) \geq \frac{l^2}{(40\log l)^2}$ for all sufficiently large l. This proves the following theorem.

Theorem 3.1 *There is a positive constant $c > 0$, such that $R(3,l) \geq c\frac{l^2}{\log^2 l}$ for all $l \geq 3$.*

So we have proved $c\frac{l^2}{\log^2 l} \leq R(3,l) \leq \frac{1}{2}l(l+1)$. We conclude this section by remarking that the true answer is $R(3,l) = \Theta\left(\frac{l^2}{\log l}\right)$. (The upper bound was proved by Ajtai, Komlós, Szemerédi and the lower bound by Jeong Han Kim.)