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Lecture 13. Algorithmic discrepancy continued

This is the second part of the analysis of the algorithm by Lovett-Meka, see Lecture 12. We are in the middle of proving Lemma 12.2.

Claim 13.1 $\mathbb{E}[(\mathbf{X}_T)_i^2] \leq 1$.

Note that by Claim 12.10 with high probability $\mathbf{X}_T \in \mathcal{P}$ and then $(\mathbf{X}_T)_i^2 \leq 1$. However, with some small probability $|(\mathbf{X}_T)_i|$ can be larger than 1 and the claim is that the total expected value of $(\mathbf{X}_T)_i^2$ is still at most 1.[1]

Proof: We use the following facts about the conditional expectations for a normal (one-dimensional) distribution $U \sim \mathcal{N}(0, \sigma^2)$ with $\sigma^2 \leq 1$: For all $\lambda \geq 0$

$$\mathbb{E}[(U - \lambda)^2 | U > \lambda] \leq 1$$

and

$$\mathbb{E}[(U - \lambda) | U > \lambda] \leq 1,$$

where the second statement is a consequence of the first using convexity.

In order to prove the claim, we may condition on $|(\mathbf{X}_T)_i| > 1 - \delta$. Otherwise we clearly have $(\mathbf{X}_T)_i^2 \leq 1$. So let us now condition on the last time that the i-th coordinate is smaller than $1 - \delta$, i.e. let $t \in \{1, \ldots, T - 1\}$ be maximal with $|(\mathbf{X}_t)_i| < 1 - \delta$. Then $|(\mathbf{X}_{t+1})_i| \geq 1 - \delta$. Let us w.l.o.g. assume $(\mathbf{X}_{t+1})_i \geq 1 - \delta$ (the case $(\mathbf{X}_{t+1})_i \leq -(1 - \delta)$ is analogous). Then let us condition on the value of $X' = (\mathbf{X}_t)_i$. As $X' = (\mathbf{X}_t)_i < 1 - \delta$, we can write $X' = 1 - \delta - \lambda\gamma$ for some $\lambda > 0$ (recall that we fixed $\gamma > 0$ in Lecture 12).

Now let X'' be a random variable defined by $X'' = X' + \gamma U = 1 - \delta - \lambda\gamma + \gamma U$, where U is the i-th standard coordinate of a Gaussian $\mathcal{N}(\mathcal{V}_t)$. By Lemma 12.7 we have $U \sim \mathcal{N}(0, \sigma^2)$ with $\sigma^2 \leq 1$. Note that the distribution of $(\mathbf{X}_{t+1})_i$ is the distribution of X'' conditioned on $X'' \geq 1 - \delta$. Note that since $(\mathbf{X}_{t+1})_i \geq 1 - \delta$ the i-th coordinate gets frozen after this step (i.e. $i \in \mathcal{C}_t^{var}$), hence $(\mathbf{X}_{t+1})_i = (\mathbf{X}_{t+2})_i = \cdots = (\mathbf{X}_T)_i$. So

$$\mathbb{E}[(\mathbf{X}_T)_i^2] = \mathbb{E}[(X'')^2 | X'' \geq 1 - \delta] = \mathbb{E}[(1 - \delta - \lambda\gamma + \gamma U)^2 | U \geq \gamma] =$$

$$= \mathbb{E}[(1 - \delta)^2 + 2(1 - \delta)\gamma(U - \lambda) + \gamma^2(U - \lambda)^2 | U \geq \gamma] =$$

$$= (1 - \delta)^2 + 2(1 - \delta)\gamma\mathbb{E}[(U - \lambda) | U \geq \gamma] + \gamma^2\mathbb{E}[(U - \lambda)^2 | U \geq \gamma] \leq$$

$$\leq (1 - \delta)^2 + 2(1 - \delta)\gamma + \gamma^2 = (1 - \delta + \gamma)^2 \leq 1,$$

where in the last line we used the assumption $\gamma < \delta$ (since we choose γ small with respect to δ, see Lecture 12). This finishes the proof of the claim. □

[1] We note that this claim does not seem to be proven very rigorously by Lovett-Meka; we fill in the details here.

Recall that in Claim 12.11 we showed that the expected number of nearly-tight discrepancy constraints satisfies $\mathbb{E}[|\mathcal{C}_T^{disc}|] \leq \frac{n}{16}$. We now prove a lower bound on the expected number of nearly-tight variable constraints.

Claim 13.2 $\mathbb{E}[|\mathcal{C}_T^{var}|] \geq \frac{3}{4}n$.

Proof: For $t = 1, \ldots, T$ we have

$$\mathbb{E}[\|\mathbf{X}_t\|^2] = \mathbb{E}[\|\mathbf{X}_{t-1} + \gamma \mathbf{U}_t\|^2] = \mathbb{E}[\|\mathbf{X}_{t-1}\|^2 + 2\gamma(\mathbf{X}_{t-1} \cdot \mathbf{U}_t) + \gamma^2\|\mathbf{U}_t\|^2] = \mathbb{E}[\|\mathbf{X}_{t-1}\|^2] + \gamma^2\mathbb{E}[\|\mathbf{U}_t\|^2],$$

where we used that $\mathbb{E}[\mathbf{U}_t|\mathbf{X}_{t-1}] = 0$ implies $\mathbb{E}[\mathbf{X}_{t-1} \cdot \mathbf{U}_t] = 0$. Since $\mathbf{U}_t \sim \mathcal{N}(\mathcal{V}_t)$, we have $\mathbb{E}[\|\mathbf{U}_t\|^2] = \mathbb{E}[\dim \mathcal{V}_t]$ and therefore

$$\mathbb{E}[\|\mathbf{X}_t\|^2] = \mathbb{E}[\|\mathbf{X}_{t-1}\|^2] + \gamma^2\mathbb{E}[\dim \mathcal{V}_t].$$

Summing this up for $t = 1, \ldots, T$ we obtain

$$\mathbb{E}[\|\mathbf{X}_T\|^2] = \mathbb{E}[\|\mathbf{X}_0\|^2] + \gamma^2 \sum_{t=1}^{T} \mathbb{E}[\dim \mathcal{V}_t] \geq \gamma^2 T\, \mathbb{E}[\dim \mathcal{V}_T],$$

using Claim 12.9 in the last step. By definition of $\mathcal{V}_T$ we have $\dim \mathcal{V}_T \geq n - |\mathcal{C}_T^{var}| - |\mathcal{C}_T^{disc}|$. Plugging this in together with $T = \frac{16}{3\gamma^2}$ yields

$$\mathbb{E}[\|\mathbf{X}_T\|^2] \geq \frac{16}{3}\mathbb{E}[n - |\mathcal{C}_T^{var}| - |\mathcal{C}_T^{disc}|] = \frac{16}{3}(n - \mathbb{E}[|\mathcal{C}_T^{var}|] - \mathbb{E}[|\mathcal{C}_T^{disc}|]).$$

On the other hand $\mathbb{E}[\|\mathbf{X}_T\|^2] = \sum_{i=1}^{n} \mathbb{E}[(\mathbf{X}_T)_i^2] \leq n$ by Claim 13.1, so we can conclude

$$n \geq \frac{16}{3}(n - \mathbb{E}[|\mathcal{C}_T^{var}|] - \mathbb{E}[|\mathcal{C}_T^{disc}|]).$$

Hence

$$\frac{3}{16}n \geq n - \mathbb{E}[|\mathcal{C}_T^{var}|] - \mathbb{E}[|\mathcal{C}_T^{disc}|]$$

and

$$\mathbb{E}[|\mathcal{C}_T^{var}|] + \mathbb{E}[|\mathcal{C}_T^{disc}|] \geq \frac{13}{16}n.$$

From Claim 12.11 we know that $\mathbb{E}[|\mathcal{C}_T^{disc}|] \leq \frac{1}{16}n$, hence $\mathbb{E}[|\mathcal{C}_T^{var}|] \geq \frac{12}{16}n = \frac{3}{4}n$. □

From the last claim we get

$$\mathbb{E}[n - |\mathcal{C}_T^{var}|] \leq \frac{n}{4}.$$

Let us apply a Markov bound to the random variable $n - |\mathcal{C}_T^{var}|$:

$$\mathbb{P}[n - |\mathcal{C}_T^{var}| \geq \frac{n}{2}] \leq \frac{\mathbb{E}[n - |\mathcal{C}_T^{var}|]}{n/2} \leq \frac{1}{2}.$$

Hence

$$\mathbb{P}[|\mathcal{C}_T^{var}| \leq \frac{n}{2}] \leq \frac{1}{2}$$

and

$$\mathbb{P}[|\mathcal{C}_T^{var}| > \frac{n}{2}] \geq \frac{1}{2}.$$

So with probability at least $\frac{1}{2}$ the point $\mathbf{X}_T$ has more than $\frac{n}{2}$ coordinates with absolute value at least $1-\delta$. By Claim 12.10 we have with high probability $\mathbf{X}_T \in \mathcal{P}$, i.e. $\mathbf{X}_T \in [-1,1]^n$ and $|\langle\langle \mathbf{X}_T - \mathbf{X}_0, \mathbf{v}_j\rangle| \leq c_j$ for all $j \in [m]$. So with probability at least $\frac{1}{3}$ we have $\mathbf{X}_T \in [-1,1]^n$, $|\langle\langle \mathbf{X}_T - \mathbf{X}_0, \mathbf{v}_j\rangle| \leq c_j$ for all $j \in [m]$, and $\mathbf{X}_T$ has more than $\frac{n}{2}$ coordinates with absolute value at least $1-\delta$. *This finishes the proof of Lemma 12.2.*

Let us now prove Theorem 12.1. by applying Lemma 12.2 repeatedly.

For $j = 1, \ldots, m$ set $\mathbf{a}_j = \chi_{A_j}$. Also recall that $\delta = \frac{1}{8\log m}$.

In the first iteration we apply Lemma 12.2 using all n coordinates, $\mathbf{x}_0 = 0$, $\mathbf{v}_j = \frac{\mathbf{a}_j}{\|\mathbf{a}_j\|} = \frac{\chi_{A_j}}{\sqrt{|A_j|}}$ and $c_j = c(m,n) = 8\sqrt{\log\frac{2m}{n}}$ for all $j \in [m]$. By the Lemma we can find (with constant probability) a point $\mathbf{x}^{(1)} \in [-1,1]^n$ such that $|\langle \mathbf{x}^{(1)}, \mathbf{a}_j\rangle| \leq c(m,n)\sqrt{|A_j|}$ for all $j \in [m]$ and $|\mathbf{x}_i^{(1)}| \geq 1-\delta$ for at least $\frac{n}{2}$ coordinates $i \in [n]$. We then set

$$I_1 = \{i \in [n] : |\mathbf{x}_i^{(1)}| < 1-\delta\}$$

and $n_1 = |I_1|$. Note that $n_1 \leq \frac{1}{2}n$. We now restrict our attention to the coordinates in I_1 and apply Lemma 12.2 again using only these n_1 coordinates, $\mathbf{x}_0 = \mathbf{x}^{(1)}|_{I_1}$, $\mathbf{v}_j = \frac{\mathbf{a}_j}{\|\mathbf{a}_j\|_{I_1}} = \frac{\mathbf{a}_j}{\sqrt{|A_j \cap I_1|}}$ and $c_j = c(m,n_1) = 8\sqrt{\log\frac{2m}{n_1}}$ for all $j \in [m]$. We find a point $\mathbf{x}^{(2)} \in [-1,1]^n$ (where we set the coordinates outside I_1 to be equal to those of $\mathbf{x}^{(1)}$) such that $|\langle \mathbf{x}^{(2)} - \mathbf{x}^{(1)}, \mathbf{a}_j\rangle| \leq c(m,n_1)\sqrt{|A_j \cap I_1|}$ for all $j \in [m]$ and $|\mathbf{x}_i^{(1)}| \geq 1-\delta$ for at least $\frac{n_1}{2}$ coordinates $i \in I_1$ (and also for all coordinates outside I_1).

We then continue iteratively. After finding a point $\mathbf{x}^{(r)}$ in the r-th iteration, we set

$$I_r = \{i \in I_r : |\mathbf{x}_i^{(r)}| < 1-\delta\} = \{i \in [n] : |\mathbf{x}_i^{(r)}| < 1-\delta\}$$

and $n_r = |I_r|$. Note that $n_r \leq \frac{1}{2}n_{r-1}$ (and so inductively $n_r \leq \frac{1}{2^r}n$). We now restrict our attention to the coordinates in I_r and apply Lemma 12.2 again using only these n_r coordinates, $\mathbf{x}_0 = \mathbf{x}^{(r)}|_{I_r}$, $\mathbf{v}_j = \frac{\mathbf{a}_j}{\|\mathbf{a}_j\|_{I_r}} = \frac{\mathbf{a}_j}{\sqrt{|A_j \cap I_r|}}$ and $c_j = c(m,n_r) = 8\sqrt{\log\frac{2m}{n_r}}$ for all $j \in [m]$. We find a point $\mathbf{x}^{(r+1)} \in [-1,1]^n$ (where we set the coordinates outside I_r to be equal to those of $\mathbf{x}^{(r)}$) such that $|\langle \mathbf{x}^{(r+1)} - \mathbf{x}^{(r)}, \mathbf{a}_j\rangle| \leq c(m,n_r)\sqrt{|A_j \cap I_r|}$ for all $j \in [m]$ and $|\mathbf{x}_i^{(r+1)}| \geq 1-\delta$ for at least $\frac{n_r}{2}$ coordinates $i \in I_r$ (and also for all coordinates outside I_r).

After $R = \mathcal{O}(\log n)$ iterations we find a point $\mathbf{x}^{(R)}$ so that for the number n_R of coordinates where $|\mathbf{x}_i^{(R)}| < 1-\delta$, we get $n_R \leq \frac{1}{2^R}n < 1$. Then all coordinates of $\mathbf{x}^{(R)} \in [-1,1]^n$ have absolute value at least $1-\delta$.

Furthermore, for all $j \in [m]$ we have (setting $\mathbf{x}^{(0)} = 0$, $n_0 = n$)

$$|\langle \mathbf{x}^{(R)}, \mathbf{a}_j\rangle| = |\sum_{r=0}^{R-1}\langle \mathbf{x}^{(r+1)} - \mathbf{x}^{(r)}, \mathbf{a}_j\rangle| \leq \sum_{r=0}^{R-1}|\langle \mathbf{x}^{(r+1)} - \mathbf{x}^{(r)}, \mathbf{a}_j\rangle| \leq \sum_{r=0}^{R-1} c(m,n_r)\sqrt{|A_j \cap I_r|} \leq$$

$$\sum_{r=0}^{R-1} c(m,n_r)\sqrt{n_r} = \sum_{r=0}^{R-1} 8\sqrt{\log\frac{2m}{n_r}}\sqrt{n_r} \leq \sum_{r=0}^{R-1} 8\sqrt{\log\frac{2m}{\frac{1}{2^r}n}}\sqrt{\frac{1}{2^r}n} =$$

$$= 8\sqrt{n}\sum_{r=0}^{R-1}\frac{\sqrt{\log(2^r\frac{2m}{n})}}{2^{r/2}} = \mathcal{O}\left(\sqrt{n\log\frac{2m}{n}}\right).$$

However, we are not quite done with proving Theorem 12.1 yet, because the coordinates of $\mathbf{x}^{(R)}$ are not only 1 and -1 (they all have absolute value at most 1 and are close to 1 or -1). So in our final step of the proof we generate $l : [n] \to \{1, -1\}$ by rounding the coordinates of $\mathbf{x}^{(R)}$: For every $i \in [n]$ we will set $l(i) = 1$ with probability $\frac{1}{2}(1 + x_i^{(R)})$ and $l(i) = -1$ with probability $\frac{1}{2}(1 - x_i^{(R)})$.

In order to bound the probability of $|l(A_j)|$ being large, we use the following concentration bound (without proof).

Lemma 13.3 (Chernoff-Hoeffding bound) *For independent random variables $Y_1, \dots, Y_n$ with $0 \leq Y_1, \dots, Y_n \leq 1$ and $\mu = \sum_{i=1}^n \mathbb{E}[Y_i]$ and for $0 < \varepsilon \leq 1$ we have*

$$\mathbb{P}\left[\sum_{i=1}^n Y_i \geq (1+\varepsilon)\mu\right] \leq e^{-\varepsilon^2\mu/3}$$

and

$$\mathbb{P}\left[\sum_{i=1}^n Y_i \leq (1-\varepsilon)\mu\right] \leq e^{-\varepsilon^2\mu/2}.$$

For each $i \in [n]$ consider a random variable Y_i given by $\mathbb{P}[Y_i = |x_i^{(R)}| - 1 + \delta] = \frac{1}{2}(1 + |x_i^{(R)}|)$ and $\mathbb{P}[Y_i = 1 + |x_i^{(R)}| + \delta] = \frac{1}{2}(1 - |x_i^{(R)}|)$. Note that for each i we have $x_i^{(R)} \in [1-\delta, 1]$ or $x_i^{(R)} \in [-1, -1+\delta]$. In the former case set $l(i) = x_i^{(R)} - Y_i + \delta$ and in the latter case set $l(i) = x_i^{(R)} + Y_i - \delta$. Note that this gives a labelling $l : [n] \to \{1, -1\}$ and in both cases we have $\mathbb{P}[l(i) = 1] = \frac{1}{2}(1 + x_i^{(R)})$ and $\mathbb{P}[l(i) = -1] = \frac{1}{2}(1 - x_i^{(R)})$ as announced above.

Furthermore, by the definition of the Y_i we have $0 \leq Y_i \leq 2 + \delta < 3$. So $\frac{1}{3}Y_1, \dots, \frac{1}{3}Y_n$ satisfy the assumptions of the Chernoff-Hoeffding bound. Also, note that $\mathbb{E}[Y_i] = \delta$ for all i.

So we have for each $j \in [m]$

$$|l(A_j)| = |\sum_{i\in A_j} l(i)| = |\sum_{i\in A_j} x_i^{(R)} - \operatorname{sgn}(x_i^{(R)})(Y_i - \delta)| \leq |\sum_{i\in A_j} x_i^{(R)}| + |\sum_{\substack{i\in A_j\\ x_i^{(R)}>0}} (Y_i - \delta)| + |\sum_{\substack{i\in A_j\\ x_i^{(R)}<0}} (Y_i - \delta)|.$$

Recall $\mathbf{a}_j = \chi_{A_j}$, so

$$|\sum_{i\in A_j} x_i^{(R)}| = |\langle \mathbf{x}^{(R)}, \mathbf{a}_j\rangle| = \mathcal{O}\left(\sqrt{n\log\frac{2m}{n}}\right).$$

Let $A'_j = \{i \in A_j | x_i^{(R)} > 0\}$. Then $\mathbb{E}[\sum_{i\in A'_j} \frac{1}{3}Y_i] = \frac{1}{3}\sum_{i\in A'_j}\mathbb{E}[Y_i] = \frac{1}{3}\delta|A'_j|$. Hence using the Chernoff-Hoeffding bound we get

$$\mathbb{P}[\sum_{i\in A'_j}(Y_i - \delta) > 6\sqrt{\log m}\sqrt{\delta n}] \leq \mathbb{P}[\sum_{i\in A'_j} Y_i > 6\sqrt{\log m}\sqrt{\delta|A'_j|} + \delta|A'_j|] =$$

$$= \mathbb{P}\left[\sum_{i \in A'_j} \frac{1}{3} Y_i > 2\sqrt{\log m}\sqrt{\delta|A'_j|} + \frac{1}{3}\delta|A'_j|\right] = \mathbb{P}\left[\sum_{i \in A'_j} \frac{1}{3} Y_i > \left(1 + 6\sqrt{\frac{\log m}{\delta|A'_j|}}\right)\frac{1}{3}\delta|A'_j|\right] \leq$$

$$\leq e^{-36\frac{\log m}{\delta|A'_j|}\frac{1}{3}\delta|A'_j|/3} = \frac{1}{m^4}.$$

Similarly for the other tail estimate:

$$\mathbb{P}\left[\sum_{i \in A'_j} (Y_i - \delta) < -6\sqrt{\log m}\sqrt{\delta n}\right] \leq \mathbb{P}\left[\sum_{i \in A'_j} Y_i < -6\sqrt{\log m}\sqrt{\delta|A'_j|} + \delta|A'_j|\right] =$$

$$= \mathbb{P}\left[\sum_{i \in A'_j} \frac{1}{3} Y_i < -2\sqrt{\log m}\sqrt{\delta|A'_j|} + \frac{1}{3}\delta|A'_j|\right] = \mathbb{P}\left[\sum_{i \in A'_j} \frac{1}{3} Y_i < \left(1 - 6\sqrt{\frac{\log m}{\delta|A'_j|}}\right)\frac{1}{3}\delta|A'_j|\right] \leq$$

$$\leq e^{-36\frac{\log m}{\delta|A'_j|}\frac{1}{3}\delta|A'_j|/2} = \frac{1}{m^6}.$$

So in total we have

$$\mathbb{P}\left[|\sum_{\substack{i \in A_j \\ x_i^{(R)} > 0}} (Y_i - \delta)| > 6\sqrt{\log m}\sqrt{\delta n}\right] = \mathbb{P}\left[|\sum_{i \in A'_j} (Y_i - \delta)| > 6\sqrt{\log m}\sqrt{\delta n}\right] \leq \frac{1}{m^4} + \frac{1}{m^6} \leq \frac{2}{m^4}.$$

Analogously we get

$$\mathbb{P}\left[|\sum_{\substack{i \in A_j \\ x_i^{(R)} < 0}} (Y_i - \delta)| > 6\sqrt{\log m}\sqrt{\delta n}\right] \leq \frac{2}{m^4}.$$

Hence with probability at least $1 - m \cdot \frac{2}{m^4}$ we have for all $j \in [m]$

$$|l(A_j)| \leq |\sum_{i \in A_j} x_i^{(R)}| + |\sum_{\substack{i \in A_j \\ x_i^{(R)} > 0}} (Y_i - \delta)| + |\sum_{\substack{i \in A_j \\ x_i^{(R)} < 0}} (Y_i - \delta)| \leq \mathcal{O}\left(\sqrt{n \log \frac{2m}{n}}\right) + 12\sqrt{\log m}\sqrt{\delta n}.$$

Using $\delta = \frac{1}{8 \log m}$, this gives (with high probability) for all $j \in [m]$

$$|l(A_j)| \leq \mathcal{O}\left(\sqrt{n \log \frac{2m}{n}}\right) + \mathcal{O}\left(\sqrt{n}\right) = \mathcal{O}\left(\sqrt{n \log \frac{2m}{n}}\right),$$

as desired. This finishes the proof of Theorem 12.1.

13.1 Related results and open questions

Before finishing this topic, we will discuss different conjectures concerning discrepancy.

Conjecture 1 (Beck-Fiala Conjecture) *Let $A_1, \ldots, A_m \subseteq [n]$ be a set system, such that each element of $[n]$ appears in at most Δ sets A_j. Then there is a labeling $l : [n] \to \{1, -1\}$ with*

$$|l(A_j)| = \mathcal{O}\left(\sqrt{\Delta}\right)$$

for all $j \in [m]$.

Beck and Fiala proved that there is a labelling $l : [n] \to \{1, -1\}$ with $|l(A_j)| \leq 2\Delta - 1$ for all j. Srinivasan proved that there is a labelling $l : [n] \to \{1, -1\}$ with $|l(A_j)| \leq \sqrt{\Delta} \log n$ for all j (and there is also an algorithm for finding such a labelling).

Vector discrepancy. Generalizing the notion of discrepancy for set systems, we can also consider vector discrepancy. Generalizing Theorem 12.1, one can show that for any vectors $\mathbf{w}_1, \ldots, \mathbf{w}_m \in \mathbb{R}^m$ with $m \geq n$ and $\|\mathbf{w}_i\|_\infty \leq 1$ for all i (i.e. all entries of all $\mathbf{w}_i$ have absolute value at most 1), there exists $l : [n] \to \{1, -1\}$ with

$$\|\sum_{i=1}^{n} l(i)\mathbf{w}_i\|_\infty < \mathcal{O}\left(\sqrt{n \log \frac{2m}{n}}\right).$$

Note we can get Theorem 12.1 as a special case of this statement by setting the j-th component of $\mathbf{w}_i$ to be 1 if $i \in A_j$ and 0 if $i \notin A_j$. Then the entries of $\sum_{i=1}^{n} l(i)\mathbf{w}_i$ are equal to $l(A_j)$. Hence $\| \sum_{i=1}^{n} l(i)\mathbf{w}_i\|_\infty$ is the discrepancy for the labelling $l : [n] \to \{1, -1\}$.

One can also generalize the theorem by Beck-Fiala that was described above (below the conjecture) to the vector setting:

Theorem 13.4 *Let $\mathbf{w}_1, \ldots, \mathbf{w}_m \in \mathbb{R}^m$ be vectors with $\|\mathbf{w}_i\|_\infty \leq 1$ and $\|\mathbf{w}_i\|_1 \leq \delta$. Then there is $l : [n] \to \{1, -1\}$ with*

$$\|\sum_{i=1}^{n} l(i)\mathbf{w}_i\|_\infty \leq 2\delta.$$

The following is a related conjecture, still open.

Conjecture 2 (Komlós conjecture) *Let $\mathbf{w}_1, \ldots, \mathbf{w}_m \in \mathbb{R}^m$ be vectors with $\|\mathbf{w}_i\|_2 \leq 1$. Then there is $l : [n] \to \{1, -1\}$ with*

$$\|\sum_{i=1}^{n} l(i)\mathbf{w}_i\|_\infty = \mathcal{O}(1).$$

Finally, let us mention Weaver's conjecture, equivalent to the famous Kadison-Singer conjecture, which was proved by Marcus, Spielman and Srivastava in 2013. On the surface, it is another type of discrepancy statement, for sums of rank 1 matrices.

Theorem 13.5 (Wearer's conjecture) *There are* $\varepsilon, \delta > 0$ *such that for all vectors* $\mathbf{v}_1, \ldots, \mathbf{v}_n \in \mathbb{C}^m$ *with*

$$\sum_{i=1}^{n} \mathbf{v}_i \mathbf{v}_i^* = I$$

and $\|\mathbf{v}_i\|_2 \leq \delta$ *for all* i*, there exists* $l : [n] \to \{1, -1\}$ *with*

$$\|\sum_{i=1}^{n} l(i) \mathbf{v}_i \mathbf{v}_i^*\| \leq \varepsilon.$$

Here, the norm refers to the spectral norm, i.e. the maximum singular value of a matrix.

We return to this topic in the last part of the course.