

Lecture 16. Stable polynomials and stability-preserving transformations

16.1 Stable polynomials

Stable polynomials generalize the notion of real-rootedness to complex variables. The following definition seems non-obvious at first, but turns out to be the right notion.

Definition 16.1 $f \in \mathbb{C}[z_1, \dots, z_n]$ is stable if $f \equiv 0$, or f has no root in \mathcal{H}^n where $\mathcal{H} = \{z \in \mathbb{C} : \text{Im}(z) > 0\}$. A stable polynomial with real coefficients is called real stable.

Lemma 16.2 If f is a univariate real stable polynomial then f is real-rooted.

Proof: Let $f(z) = \sum_{k=0}^d a_k z^k$ where $a_k \in \mathbb{R}$. Then the roots of f are either real or appear in conjugate pairs $\lambda, \bar{\lambda}$. Hence, $f(z) \neq 0 \forall z \in \mathcal{H} \Leftrightarrow f(z) \neq 0 \forall z \in -\mathcal{H}$. Thus, f must be real-rooted. \square

Note that this is not true for more than one variable, for example: $f(z_1, z_2) = 1 - z_1 z_2$ is stable but not real-rooted (as its roots are pairs (z_1, z_2) such that $z_2 = \frac{z_1}{|z_1|^2}$, which cannot be both in \mathcal{H}).

Lemma 16.3 $f \in \mathbb{C}[z_1, \dots, z_n]$ has no roots in \mathcal{H}^n iff for all $x \in \mathbb{R}^n, d \in \mathbb{R}_+^n, g(t) = f(x + td)$ (in variable $t \in \mathbb{C}$) has no zeros in \mathcal{H} .

Proof: If f is not stable, $f(z_1, \dots, z_n) = 0$ for some $z_1, \dots, z_n \in \mathcal{H}$. Then take $x_i = \text{Re}(z_i), d_i = \text{Im}(z_i)$; we have $g(i) = f(x + id) = 0$ so g has a zero in \mathcal{H} .

If $g(t) = 0$ for $t \in \mathcal{H}$ then $f(x_1 + td_1, \dots, x_n + td_n) = 0$ and $x_i + td_i \in \mathcal{H}$ since $\text{Im}(x_i + td_i) = d_i \cdot \text{Im}(t) > 0$ since $t \in \mathcal{H}$ and $d \in \mathbb{R}_+^n$. Hence, f is not stable. \square

Note: It is not correct to say that $f(z)$ is stable iff every $g(t) = f(x + td)$ as above is stable, since $f(z_1, z_2) = z_1 - z_2$ is not stable (having root (i, i)) but $g(t) = f(x_1 + td_1, x_2 + td_2) = (x_1 - x_2) + t(d_1 - d_2)$ is either real-rooted or a constant polynomial, hence, by our definition and convention, g is always stable.

Lemma 16.4 Let $f_1, f_2, \dots \in \mathbb{C}[z_1, \dots, z_n]$ be a sequence of polynomials of bounded degree with no zeros in \mathcal{H}^n and assume that $f_i \rightarrow f$ coefficient-wise. Then either $f \equiv 0$ or f has no zeros in \mathcal{H}^n .

Proof: We will prove this by induction on n . We prove the case of $n = 1$ in the previous lecture (noting that \mathcal{H} is open).

Now assume that $a \in \mathcal{H}$ is fixed. Then $f_i(z_1, \dots, z_{n-1}, a) \rightarrow f(z_1, \dots, z_{n-1}, a)$ and by the induction hypothesis it must be that either

1. $f(z_1, \dots, z_{n-1}, a) \equiv 0$ or
2. $f(z_1, \dots, z_{n-1}, a)$ has no roots in \mathcal{H}^{n-1} .

If 1 happens for all $a \in \mathcal{H}$, then $f \equiv 0$ and we are done. Similarly if 2 happens for all $a \in \mathcal{H}$, then f has no zeros in \mathcal{H}^n and we are done.

But what if 1 happens for some (but not all) $a \in \mathcal{H}$? Look at $g(z) = f(i, i, \dots, i, z) \in \mathbb{C}[z]$. Since $f(i, \dots, i, z) = \lim_{m \rightarrow \infty} f_m(i, \dots, i, z)$, we have $g(z) = 0$ for some (but not all) $z \in \mathcal{H}$. This means that $g \not\equiv 0$, but g has zeros in \mathcal{H} . This is in contradiction with continuity of the roots of univariate polynomials, as none of the $f_m(i, \dots, i, z)$ have a root in \mathcal{H} . \square

16.2 Stability from PSD matrices

A natural and interesting class of stable polynomials can be derived from PSD matrices. Recall the following definitions about Hermitian and PSD matrices.

Definition 16.5 For a matrix $A \in \mathcal{M}^n(\mathbb{C})$, define A^* to be the matrix where $(A^*)_{ij} = \overline{A_{ji}}$.

Definition 16.6 A matrix $H \in \mathcal{M}^n(\mathbb{C})$ is Hermitian iff $H^* = H$.

Definition 16.7 A Hermitian matrix H is PSD (positive semidefinite), or $H \succeq 0$, iff for all $x \in \mathbb{C}^n$ we have $x^* H x \geq 0$.

Definition 16.8 A Hermitian matrix H is PD (positive definite), or $H \succ 0$, iff for all $x \neq 0 \in \mathbb{C}^n$ we have $x^* H x > 0$.

The following lemma describes one of the most important families of real stable polynomials.

Lemma 16.9 Given PSD matrices A_1, \dots, A_n and Hermitian B , the polynomial

$$f(z_1, \dots, z_n) = \det\left(\sum_i z_i A_i + B\right)$$

is real stable.

Proof: We can assume w.l.o.g. that $A_1, \dots, A_n \succ 0$, since all PSD matrices are limits of PD matrices; real stability follows from taking limits and lemma 16.4.

We will use lemma 16.3. Let $x \in \mathbb{R}^n$ and $d \in \mathbb{R}_+^n$ and consider the polynomial

$$g(t) = \det\left(t \overbrace{\sum_i d_i A_i}^{M \succ 0} + \sum_i x_i A_i + B\right).$$

Note that the matrix $M = \sum_i d_i A_i$ is positive definite. We can therefore define the square root and inverse square roots for it: decompose M as $U^* D U$, where U is a unitary matrix and D is diagonal with positive elements. Then

$$\begin{aligned} M^{\frac{1}{2}} &= U^* D^{\frac{1}{2}} U, \\ M^{-\frac{1}{2}} &= U^* D^{-\frac{1}{2}} U. \end{aligned}$$

Now we have

$$\begin{aligned}
g(t) &= \det(tM + \sum_i x_i A_i + B) = \det(M^{\frac{1}{2}}(tI + M^{-\frac{1}{2}}(\sum_i x_i A_i + B)M^{-\frac{1}{2}})M^{\frac{1}{2}}) \\
&= \det(M) \underbrace{\det(tI + \overbrace{M^{-\frac{1}{2}}(\sum_i x_i A_i + B)M^{-\frac{1}{2}}}^{\text{Hermitian}})}_{\text{characteristic polynomial}}.
\end{aligned}$$

Therefore g is a multiple of the characteristic polynomial of a Hermitian matrix, which means that it has real roots, i.e. no roots in \mathcal{H} . \square

The class of polynomials in lemma 16.9 is usually the basis for deriving other real stable polynomials, by applying real stability preserving transformations that will be discussed shortly. However this class is rich enough that for univariate and bivariate polynomials, no other transformation is needed as witnessed by the following theorem.

Theorem 16.10 (Helton-Vinnikov) *A polynomial $p \in \mathbb{C}[z_1, z_2]$ is real stable iff p can be written as $k \cdot \det(z_1 A + z_2 B + C)$ where $k \in \mathbb{R}$ and A, B, C are real symmetric matrices where $A, B \succeq 0$.*

16.3 Operators preserving real stability

We have an important class of stable polynomials, but we can apply transformations to generate more.

Theorem 16.11 *The following transformations on $\mathbb{C}[z_1, \dots, z_n]$ preserve stability.*

1. **Permutation:** $f(z_1, \dots, z_n) \mapsto f(z_{\sigma(1)}, \dots, z_{\sigma(n)})$ for some $\sigma \in S_n$.
2. **Scaling:** $f(z_1, \dots, z_n) \mapsto f(az_1, \dots, z_n)$ for $a \in \mathbb{R}_+$.
3. **Diagonalization:** $f(z_1, z_2, \dots, z_n) \mapsto f(z_2, z_2, z_3, \dots, z_n)$.
4. **Inversion:** $f(z_1, \dots, z_n) \mapsto z_1^d f(-\frac{1}{z_1}, \dots, z_n)$ where $d = \deg_{z_1} f$.
5. **Specialization:** $f(z_1, \dots, z_n) \mapsto f(a, z_2, \dots, z_n)$ for some $a \in \mathcal{H} \cup \mathbb{R}$.
6. **Differentiation:** $f(z_1, \dots, z_n) \mapsto \frac{\partial}{\partial z_1} f(z_1, \dots, z_n)$.

Proof: Let us go through the operations, one by one.

Permutation: Permutation on coordinates is a bijection between \mathcal{H}^n and itself.

Scaling: Scaling by a constant $a \in \mathbb{R}_+$ is also a bijection between \mathcal{H} and itself.

Diagonalization: By definition $f(z_2, z_2, \dots, z_n) \neq 0$ for $z_2, \dots, z_n \in \mathcal{H}$.

Inversion: The map $z_1 \mapsto -\frac{1}{z_1}$ is a bijection between \mathcal{H} and itself.

Specialization: Immediate as by definition $f(a, z_2, \dots, z_n) \neq 0$ for $a, z_2, \dots, z_n \in \mathcal{H}$.

Differentiation: This follows immediately from the Gauss-Lucas theorem which states that for $f \in \mathbb{C}[z]$, the roots of f' lie in the convex hull of the roots of f (note that the complement of \mathcal{H} is convex). Next we will see a proof of the Gauss-Lucas theorem. \square

Theorem 16.12 (Gauss-Lucas) *If $f \in \mathbb{C}[z]$, then the roots of f' are in $\text{conv}(\text{roots of } f)$.*

Proof: Let $\lambda_1, \dots, \lambda_n$ be the roots of f and assume that $f'(z) = 0$ for some z . If $f(z) = 0$, we are done, so assume that $f(z) \neq 0$. Therefore

$$0 = \frac{f'(z)}{f(z)} = \sum_i \frac{1}{z - \lambda_i} = \sum_i \frac{\overline{z - \lambda_i}}{|z - \lambda_i|^2}.$$

We can rewrite this as

$$\sum_i \frac{z}{|z - \lambda_i|^2} = \sum_i \frac{\lambda_i}{|z - \lambda_i|^2},$$

or equivalently

$$z = \left(\sum_i \frac{1}{|z - \lambda_i|^2} \right)^{-1} \sum_i \frac{\lambda_i}{|z - \lambda_i|^2},$$

which shows that z is a convex combination of $\lambda_1, \dots, \lambda_n$. □

Let us now apply some of these operations to obtain a combinatorial fact as an exercise.

Example 16.13 (Spanning tree polynomial) For a graph $G = (V, E)$ define the spanning tree polynomial $\tau_G \in \mathbb{C}[z_e : e \in E]$ as

$$\tau_G(z_e : e \in E) = \sum_{\substack{T \subseteq E \\ T \text{ is a spanning tree}}} \prod_{e \in T} z_e.$$

Lemma 16.14 The polynomial τ_G is real stable.

Proof: We use the matrix-tree theorem. Let L_G be the Laplacian matrix of the graph G defined by

$$(L_G)_{ii} = \deg(i),$$

$$(L_G)_{ij} = \begin{cases} -1 & \text{if } (i, j) \in E, \\ 0 & \text{otherwise.} \end{cases}$$

For an edge $e = (i, j)$ let $b_e = \delta_i - \delta_j$ where δ_i is i -th element of the standard basis in \mathbb{R}^n . In other words let

$$b_e = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ \vdots \\ -1 \\ \vdots \\ 0 \\ 0 \end{pmatrix},$$

where the only nonzero entries are the i -th and j -th entries. It is easy to see that

$$L_G = \sum_{e \in E} b_e b_e^\top.$$

The matrix-tree theorem states that the number of spanning trees of G is equal to $\det(\widetilde{L}_G)$ where \widetilde{L}_G is obtained from L_G by removing the first row and column. If we let \widetilde{b}_e be the vector obtained from b_e by removing the first component, then $\widetilde{L}_G = \sum_{e \in E} \widetilde{b}_e \widetilde{b}_e^\top$. More generally one can see that

$$\tau_G(z_e : e \in E) = \det\left(\sum_{e \in E} z_e \underbrace{\widetilde{b}_e \widetilde{b}_e^\top}_{\succeq 0}\right).$$

From lemma 16.9, it follows that the polynomial τ_G is real stable. □

We will prove the following combinatorial fact using the tools we have covered.

Lemma 16.15 *Let T be a uniformly random spanning in the graph $G = (V, E)$. Assume that $F \subseteq E$ is fixed. Then $|T \cap F|$ has the same distribution as the sum of some independent $\{0, 1\}$ -valued random variables.*

Proof: Let $Q_F \in \mathbb{R}[z]$ be defined as

$$Q_F(z) = \mathbb{E}[z^{|T \cap F|}] = \sum_{k=0}^{|F|} q_k z^k,$$

where $q_k = \Pr[|T \cap F| = k]$.

But note that Q_F can be obtained from τ_G in the following way:

$$Q_F(z) = \frac{1}{\text{number of spanning trees}} \tau_G(\underbrace{z, \dots, z}_F, 1, \dots, 1).$$

Therefore Q_F is real stable. Since Q_F is univariate this means that Q_F has real roots. Since Q_F has positive coefficients, these roots must be negative. Any number in \mathbb{R}_- can be written as $-\frac{p}{1-p}$ for some $p \in (0, 1)$. Therefore we can write

$$Q_F(z) = c \cdot \prod_i \left(z + \frac{p_i}{1-p_i}\right),$$

or equivalently

$$Q_F(z) = k \cdot \prod_i ((1-p_i)z + p_i),$$

for $k, c \in \mathbb{R}$ and $p_i \in (0, 1)$.

Note that $k = Q_F(1) = 1$. Therefore

$$Q_F(z) = \prod_i ((1-p_i)z + p_i).$$

This is the generating polynomial of the sum of independent $\{0, 1\}$ -valued random variables with the probability of being 0 given by p_i . □

16.4 Characterization of stability preserving transformations

We have seen some examples operations that preserve real stability. The following powerful theorem characterizes all such transformations that are linear, which includes the ones we saw in theorem 16.11.

Definition 16.16 Let $\mathbb{C}_k[z_1, \dots, z_n]$ be the subset of $\mathbb{C}[z_1, \dots, z_n]$ consisting of polynomials of degree at most k .

Theorem 16.17 (Borcea-Brändén) Assume the linear operator $T : \mathbb{C}_k[z_1, \dots, z_n] \rightarrow \mathbb{C}[z_1, \dots, z_n]$ is non-degenerate or in other words $\dim(\text{range}(T)) \geq 2$.

Then T preserves stability iff the polynomial $G_T \in \mathbb{C}[z_1, \dots, z_n, w_1, \dots, w_n]$ defined below is stable.

$$G_T(z_1, \dots, z_n, w_1, \dots, w_n) = T[(z_1 + w_1)^k \dots (z_n + w_n)^k]$$

Remark 16.18 In the above theorem when T is applied to a polynomial with both z_i 's and w_i 's, the variables w_1, \dots, w_n are treated as constants (i.e. they commute with T).