Stanford University	**Spring 2016**
Math 233: Non-constructive methods in combinatorics	**Instructor: Jan Vondrák**
Lecture date: May 23, 2016	**Scribe: Alex Dunlap**

Lecture 17. Ramanujan graphs from interlacing polynomials

Here we follow the paper *Interlacing families I: Bipartite Ramanujan graphs of all degrees* by Marcus, Spielman and Srivastava.

17.1 Ramanujan graphs

What is a Ramanujan graph? Generally people are very interested in expanders: graph G, A_G adjacency matrix. Suppose G is d-regular. Then d is always an eigenvalue of A_G with eigenvector $\mathbf{1}$. The remaining eigenvalues tell you a lot about the structure of the graph. In particular, if all of the other eigenvalues are significantly smaller in absolute value than d, then G is a "good expander." This means that for any set of vertices S, $|S| \leq n/2$, there are many edges leaving S — proportional to $d|S|$. Graphs with all eigenvalues small except for d are in fact *pseudorandom* — e.g., if you partition the vertices into two sets $S, \overline{S}$, then you have roughly the number of edges between them that you would expect in a random graph. Ramanujan graphs are in some sense the *best possible* expanders.

Definition 17.1 *A graph G is Ramanujan if it's d-regular and its eigenvalues, apart from one eigenvalue d and possibly one eigenvalue $-d$ (if bipartite), are in the interval $[-2\sqrt{d-1}, 2\sqrt{d-1}]$.*

It is known [Alon-Boppana 1991] that $2\sqrt{d-1}$ is best-possible for any infinite family of d-regular graphs. That is, for any d and any $\epsilon > 0$, there are only finitely many d-regular graphs whose eigenvalues apart from $\{-d, d\}$ lie inside $[-2\sqrt{d-1}+\epsilon, 2\sqrt{d-1}-\epsilon]$.

Example 17.2 *The eigenvalues of K_{d+1} are d with multiplicity 1 and -1 with multiplicity d. So it is a good expander in some sense, but we want to keep d small compared to $n = V(G)$.*

Example 17.3 *The eigenvalues of $K_{d,d}$ are d and $-d$ each with multiplicity 1 and 0 with multiplicity $2(d-1)$.*

The first explicit constructions of Ramanujan graphs were found by [Lubotsky–Philips–Sarnak '88] and [Margulis '88]. These are (infinite families of) Ramanujan graphs for $d = p+1$, p prime. These graphs are non-bipartite, so there is a single eigenvalue d and all the others have absolute value at most $2\sqrt{d-1}$. The result that we will see in this lecture is as follows.

Theorem 17.4 (MSS '13) *For any $d \geq 3$, there is an infinite family of* bipartite d-regular *Ramanujan graphs.*

[Bilu-Linial '06] proposed an iterative construction which forms the basis of the approach of [MSS '13]. How does this work? We want to take an expander and get an expander on twice the number of vertices. Given connected vertices u and v, we add vertices u_1 and v_1 and either connect (u, v) and (u_1, v_1) or connect (u, v_1) and (u_1, v). This is called a 2-*lift of G, denoted $G_s^{(2)}$*. It is determined by $s_e \in \{-1, 1\}$ for each edge E. (Bilu and Linial proved that their approach gives graphs with non-trivial eigenvalues bounded by $|\lambda| \leq O(\sqrt{d \log^3 d})$.)

Lemma 17.5 *If A is the adjacency matrix of G, and A_s is the signed adjacency matrix, where we put a sign for each edge e according to s_e, then the eigenvalues of the adjacency matrix of $G_s^{(2)}$ are the union of the eigenvalues of A and the eigenvalues of A_s.*

Proof: If $Ax = \lambda x$ for some $x \in \mathbf{R}^V$, then we can define $x^{(2)} \in \mathbf{R}^{2|V|}$ in which we duplicate each coordinate. Then if $Ax = \lambda x$ then $A_s^{(2)} x^{(2)} = \lambda x^{(2)}$. Define $x^{(-)} \in \mathbf{R}^{2|V|}$ in which we duplicate each coordinate, but with the second one negative. Then if $A_s x = \mu x$, then $A_s^{(2)} x^{(-)} = \mu x^{(-)}$. □

The goal is to control the eigenvalues of A_s, which are the roots of the characteristic polynomial $\chi_s(x) = det(xI - A_s)$. Interestingly, it was already known that the *expectation* of $\chi_s(x)$ over random signings $s \in \pm 1^E$ is exactly the matching polynomial of G.

Lemma 17.6 (Godsil–Gutman) *Let $\chi_s(x) = \det(xI - A_s)$. Then*

$$\mathbf{E}_{s \in \{\pm 1\}^E} \chi_s(x) = \mathcal{M}_G(x).$$

Proof: Write

$$\chi_s(x) = \sum_{\sigma \in S_n} (-1)^\sigma x^{\#\{\text{fixed points of } \sigma\}} \prod_{\sigma(i) \neq i} (-s_{i,\sigma(i)}).$$

Then

$$\mathbf{E}_s \chi_s(x) = \sum_{\sigma \in S_n} (-1)^\sigma x^{\#\{\text{fixed points of } \sigma\}} \mathbf{E}_s \left[\prod_{\sigma(i) \neq i} (-s_{i,\sigma(i)}) \right].$$

Whenever you have an entry $(i, \sigma(i))$ such that the $(\sigma(i), i)$ does not appear in the product, then that coordinate is independent of everything else, so you get 0. So the expectation is 0 unless you have $(\sigma(i), i)$ whenever you have $(\sigma(i), i)$; otherwise it's 1. Thus you only have to consider permutations whose only cycles are of length 1 and 2. Then the last quantity becomes exactly the matching polynomial (since $(-1)^\sigma$ is the number of transpositions to make the permutation). □

Heilmann–Lieb tells us that the maximum root of $\mathcal{M}_G(x)$ is at most $2\sqrt{d-1}$. So we would be done, if the roots of $\chi_s(x)$ for some particular s were bounded by the maximum root of $\mathbb{E}[\chi_s(x)]$. This is not true in general, but in this particular scenario it happens to be true! This is exactly what the method of interlacing polynomials accomplishes.

Definition 17.7 *$f, g \in \mathbf{R}[x]$ of degree n have a common interlacing if they are real-rooted and there is a polynomial h so that both f and g interlace the polynomial h (equivalently, the roots of f and g alternate). (Note that f and g may have repeated roots.)*

Lemma 17.8 *f and g (real-rooted) have a common interlacing if and only if for all $t \in [0,1]$, $h_t(x) = tf(x) + (1-t)g(x)$ is real-rooted.*

Proof: ($\Longleftarrow$) As t goes from 0 to 1, the roots of h_t move continuously on the real line. Assume first that f and g have no shared roots. Put $\lambda_k(t)$ to be the kth largest root of h_t. We will show that $I_k = (\lambda_k(0), \lambda_k(1))$ are disjoint by proving the following claim: for $t \in (0,1)$, $\lambda_k(t)$ cannot be a root of f or of g. This is true because $0 = h_t(\lambda_k(t)) = tf(\lambda_k(t)) + (1-t)g(\lambda_k(t))$, so if $f(\lambda_k(t)) = 0$ would imply that $g(\lambda_k(t)) = 0$, so they would have a shared root. If f and g have shared roots, then factor out the shared root and apply the argument to the remaining polynomials, and then at

the end add the shared roots back in. If the roots fall into one of the I_ks, you have to split up the interval and be careful about assigning roots to each side.

($\Longrightarrow$) Assume f and g have a common interlacing. Then between $\gamma_n, \gamma_{n-1}, \ldots$ (points separating the roots of f and g), f and g alternate signs (in the same fashion). (Assume roots are simple, $\gamma_n < \gamma_{n-1} < \cdots < \gamma_1 < \gamma_0$.) Then $h_t(x)$ has a root in each interval (γ_i, γ_{i-1}). Thus h_t has n distinct roots. In general, take perturbations $f_\varepsilon, g_\varepsilon$ with simple roots and apply the theorem to those. □

The following lemma follows from the same argument.

Lemma 17.9 *If λ_k is the kth largest root of f, and λ'_k is the kth largest root of g, and $\mu_k^{(t)}$ is the kth largest root of h_t, then* $\min\{\lambda_k, \lambda'_k\} \leq \mu_k^{(t)} \leq \max\{\lambda_k, \lambda'_k\}$.