MATH 233A: Concentration of Measure, Autumn 2018
HOMEWORK 1
Due Monday, October 29

Please try to solve the homework on your own. Discussions are okay but make your own effort.

Problem 1. Assume that $Z = f(X_1, \dots, X_n)$ where $X_1, \dots, X_n$ are independent Bernoulli ($\{0,1\}$) variables with expectations $p_1, \dots, p_n$, and changing X_i can affect the value of Z by at most c_i. Prove that

$$\mathrm{Var}[Z] \leq \sum_{i=1}^{n} c_i^2 p_i (1 - p_i).$$

Problem 2. Let Z be the number of triangles in the random graph $G_{n,p}$. Compute $\mathrm{Var}[Z]$, and compute the upper bound given by the Efron-Stein inequality (where the random variables X_i are indicators of the edges).

Problem 3. Let $X_1, \dots, X_n$ be (correlated) random variables. Let $\alpha_1, \dots, \alpha_k \in [0,1]$ and $S_1, \dots, S_k \subseteq [n]$ be such that for each $j \in [n]$, $\sum_{i: j \in S_i} \alpha_i = 1$. Prove that

$$H(X_1, \dots, X_n) \leq \sum_{i=1}^{k} \alpha_i H(X_j : j \in S_i).$$

Problem 4. Let A be a finite subset of $\mathbb{Z}^d$. For $1 \leq i \leq d$, let

$$A^{(i)} = \{(x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n) : (x_1, \dots, x_n) \in A\}.$$

Use information theory to prove that

$$\prod_{i=1}^{d} |A^{(i)}| \geq |A|^{d-1}.$$

Problem 5. Prove that for a fixed distribution P, the KL divergence $D(Q \| P)$ is a convex functional of Q (for any $\lambda \in [0,1]$, $D(\lambda Q + (1-\lambda) Q' \| P) \leq \lambda D(Q \| P) + (1-\lambda) D(Q' \| P)$).

Bonus Problem. (I don't know the solution at the moment...)
Let $Z = f(X_1, \dots, X_n)$ where $X_1, \dots, X_n$ are independent. Let $Z_i = \inf_{X_i} f(X_1, \dots, X_n)$ and $V = \sum_{i=1}^{n} (Z - Z_i)^2$. Assume that W is a random variable such that $V \leq WZ$ with probability 1. Then prove that

$$\mathrm{Var}[\sqrt{Z}] \leq \mathbb{E}[W].$$