Stanford University	**Spring 2018**
Math 233A: Non-constructive methods in combinatorics	**Instructor: Jan Vondrák**
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Introduction

Our two main topics will be the Lovász Local Lemma and the method of interlacing polynomials (stable polynomials). Both of these can be thought of as methods involving polynomials. In the Lovász Local Lemma, the relevant polynomial is the independence polynomial of a graph, defined as

$$I_G(p_1, \dots, p_n) = \sum_{I \in \mathrm{Ind}(G)} \prod_{i \in I} p_i$$

In the method of stable polynomials, especially in its application to the Kadison-Singer problem, the relevant polynomial is the characteristic polynomial of a matrix,

$$P_A(\lambda) = \det(\lambda I - A)$$

Both of these methods are non-constructive, and a natural question is whether they can be made algorithmic (or constructive)—can we find the objects that these objects prove exist? For the Lovász Local Lemma, the answer is more or less yes. However, for the method of stable polynomials, we don't really know.

1.1 Hypergraph 2-coloring

A hypergraph is just a set system: there is some ground set V, and the hypergraph is just some collection of subsets, $\mathcal{F} \subseteq 2^V$. The goal is to color the elements of V red and blue so that no element of $\mathcal{F}$ is monochromatic.

Let's assume that the hypergraph is k-uniform, namely that all sets in $\mathcal{F}$ have cardinality k. One natural approach is to color randomly: we independently assign each vertex the colors red and blue with probability $1/2$. Then for a set $A_i \in \mathcal{F}$,

$$\Pr(A_i \text{ is monochromatic}) = \frac{2}{2^k}$$

So if $|\mathcal{F}| = m$, then the union bound says that

$$\Pr(\text{some } A_i \in \mathcal{F} \text{ is monochromatic}) \leq \frac{2m}{2^k}$$

Thus, if $m < 2^{k-1}$, then this probability is strictly less than 1, and thus there is a legal 2-coloring.

The Lovász Local Lemma (LLL) gives a stronger result. For instance, let's assume that k is a constant, m is huge, but that the interactions between the sets is somewhat limited. More specifically, say we assume that each set in $\mathcal{F}$ intersects at most d other sets, for some parameter d. Then the LLL implies that if $d < 2^{k-1}/e - 1$, then there exists a legal 2-coloring, regardless of m.

1.2 The Kadison-Singer Problem

The problem of hypergraph 2-coloring can be thought of as a combinatorial balancing problem: we want to partition a space into two sets such that a certain combinatorial balancing requirement is met. Analogously, the Kadison-Singer problem is a *spectral* balancing problem.

Let $u_i \in \mathbb{R}^n$ be a collection of m vectors, with the property that their outer products sum to the identity:

$$I = \sum_{i=1}^{m} u_i u_i^T$$

Assume also that the u_i are small, namely that $\|u_i\| \leq \delta$ for some parameter δ. The goal is to partition $[m]$, the indexing set, as $[m] = S_1 \dot{\cup} S_2$ so that

$$\left\| \sum_{i \in S_1} u_i u_i^T \right\| \leq \frac{1}{2} + \varepsilon \quad \text{and} \quad \left\| \sum_{i \in S_2} u_i u_i^T \right\| \leq \frac{1}{2} + \varepsilon$$

where this norm is the spectral norm (in this case simply the largest eigenvalue of the matrix). Equivalently, this requires us to ensure that the eigenvalues of $\sum_{i \in S_1} u_i u_i^T$ are in $[\frac{1}{2} - \varepsilon, \frac{1}{2} + \varepsilon]$.

Another way of thinking about this problem is as follows: for any unit vector x, we have that

$$1 = x^T I x = \sum_{i=1}^{m} x^T u_i u_i^T x = \sum_{i=1}^{m} (u_i^T x)^2$$

This property, namely $\sum u_i u_i^T = I$, is called being in *isotropic position*, for exactly this reason: the above calculation shows that for any direction x, the variance of the projection of the point set onto the line defined by x is 1. In particular, all lines behave the same. From this perspective, the Kadison-Singer problem asks us to color the points red and blue so that the red and blue sets are also pretty close to isotropic; indeed, the property of all the eigenvalues being in $[\frac{1}{2} - \varepsilon, \frac{1}{2} + \varepsilon]$ precisely implies such a near-isotropic distribution.

The Kadison-Singer problem was recently (circa 2013) solved by Marcus, Spielman, and Srivastava, and they developed the method of stable polynomials for this problem.

1.3 Ramsey's Theorem

Theorem 1 (Ramsey) *For any t, there exists some large enough $n = n(t)$ such that for any 2-edge-coloring of K_n, there is a monochromatic K_t.*

A natural question is how large n has to be as a function of t. A simple inductive proof gives the upper bound of $n(t) \leq 4^t$. For the lower bound, the simplest thing to do is to make $t-1$ blocks of $t-1$ vertices, color all block-internal edges red, and color all edges between blocks blue. Then this will definitely not have a monochromatic K_t, by the pigeonhole principle. Thus, $n(t) > (t-1)^2$. This is what we would call an "explicit" construction (namely some sort of closed formula), and it's hard to find explicit constructions that do much better. More general than an explicit construction is a deterministic polynomial-time algorithm that outputs the coloring, and even more general is a polynomial-time randomized algorithm, and with success probability at least inverse polynomial in the parameters. Even more general is any finite-time algorithm.

A randomized algorithm can do much better: if we just independently color each edge red and blue with probability 1/2, then for any clique A of size t,

$$\Pr(A \text{ is monochromatic}) = \frac{2}{2^{\binom{t}{2}}}$$

The number of cliques is $\binom{n}{t}$, so by the union bound,

$$\Pr(\text{there is a monochromatic clique}) \leq \binom{n}{t} \frac{2}{2^{\binom{t}{2}}} \leq \frac{n^t}{t!} \frac{2}{2^{t^2/2 - t/2}}$$

So if we choose $n = 2^{t/2}$, then this probability is bounded by

$$\frac{2^{t^2/2}}{t!} \frac{2}{2^{t^2/2 - t/2}} = \frac{2^{1+t/2}}{t!} < 1$$

for all but maybe some small values of t. Thus, we can conclude that $n(t) > 2^{t/2}$, and this is still more or less the best known lower bound.

1.4 Non-constructive methods and making them constructive

A fundamental question that we will discuss here is whether and when existential proofs can be turned into efficient algorithms. Very roughly, the current state of affairs for different methods of existential proofs can be summarized as follows.

Method	**Constructive?**
Probabilistic (the probabilistic method, LLL, ...)	Yes
Algebraic (stable polynomials, Nullstellensatz, ...)	???
Topological (Brouwer, Borsuk-Ulam, ...)	In some cases, provably no! (existence of Nash equilibria)