Stanford University	**Spring 2018**
Math 233A: Non-constructive methods in combinatorics	**Instructor: Jan Vondrák**
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Lecture 10. Correlation Decay for Shearer's polynomials

1 Introduction

In this lecture, we are interested in the following question:

Question 1 *Can we compute efficiently the values of the independence polynomial*

$$q_\emptyset(p) = \sum_{I \in \mathsf{Ind}(G)} (-1)^{|I|} p^I ?$$

Generally, computing this polynomial is #P-hard, meaning we don't expect to do this computation *exactly* in less than exponential time. But maybe we can compute this approximately to a reasonable degree in a reasonable amount of time. A method to approximately compute q efficiently is the goal of this lecture.

2 A Recursive Approximation

Throughout, assume that p is in Shearer's region, S is a set of vertices containing u, and that u has neighbors $\{v_1, \ldots, v_d\}$ in S.

One possible strategy would be to employ the recurrence we proved in Shearer's Lemma, that

$$\breve{q}_S = \breve{q}_{S-u} - p_u \breve{q}_{S \setminus \Gamma^+(u)} \qquad \forall u \in S.$$

Running a recursive algorithm on this expression still gives an exponential worst-case running time. Additionally, there is no hope to only explore a local neighborhood of a single vertex u with this recurrence to run an approximate calculation, because the quantity $\breve{q}_S$ in some sense indeed depends on all of S. As in the proof of Shearer's Lemma, however, we can rearrange the recurrence to obtain a telescoping product

$$\frac{\breve{q}_S}{\breve{q}_{S-u}} = 1 - p_u \frac{\breve{q}_{S\setminus\Gamma^+(u)}}{\breve{q}_{S-u}} = 1 - p_u \frac{\breve{q}_{S\setminus\{u,v_1\}}}{\breve{q}_{S-u}} \frac{\breve{q}_{S\setminus\{u,v_1,v_2\}}}{\breve{q}_{S\setminus\{u,v_1\}}} \cdots$$

Definition 2.1 *For a set of vertices V and vertex v, let*

$$r_{S,u} := 1 - \frac{\breve{q}_S}{\breve{q}_{S-u}}.$$

Remark 2.2 *Letting $S_i := S \setminus \{u, v_1, \ldots, v_{i-1}\}$, we can write $r_{S,u}$ as*

$$r_{S,u} = p_u \cdot \frac{1}{1 - r_{S_1,v_1}} \cdot \frac{1}{1 - r_{S_2,v_2}} \cdots \frac{1}{1 - r_{S_d,v_d}}.$$

(If the set of neighbors is empty, $r_{S,u} = p_u$.)

Representing the recursive structure of the independence polynomial in this form, there is now a clear way to cut off the recursive tree from vertex u at some pre-specified depth. Why would cutting off the recursive tree after some level give a good approximation? We can think of $r_{S,u}$ as $\Pr\left[\mathcal{E}_u | \bigcap_{v \in S-u} \overline{\mathcal{E}}_v\right]$ in the tight instance of Shearer's lemma. Intuitively, this is claiming that somehow this conditional probability shouldn't depend too much on events that are "far away" from it in the dependency graph, so cutting them out of our computation shouldn't affect it too much.

It turns out that $r_{V,v}$ as a function of $p = (p_1, \ldots, p_n)$ is convex and non-decreasing, which will be useful later on.

Lemma 2.3 *For p in Shearer's region,*

(i) $r_{S,u}(p)$ is increasing in p

(ii) $r_{S,u}(\lambda p)$ is convex in λ.

Proof: This is proved inductively, on the size of S. For the base case, $|S| = 1$: $r_{\{u\},u}(\lambda p) = \lambda p_u$, which is non-decreasing in each p_i and convex in λ. For the inductive step, assume the $r_{S,u}$'s are increasing and convex for $|S| \leq k$. On the left hand side of the recursive formula, letting $|S| = k+1$, this means on the right hand side of the recursive formula, all the r's that appear have corresponding sets of size $\leq k$. By the inductive hypothesis, these are increasing and convex. Since $1/(1-x)$ is an increasing function of x for $x \in [0, 1)$ and all r's are between 0 and 1, $p_u \prod_{i=1}^{d} \frac{1}{1-r_{S_i,v_i}(p)}$ is increasing in p. Similarly, by considering the derivative with respect to λ, $\lambda p_u \prod_{i=1}^{d} \frac{1}{1-r_{S_i,v_i}(\lambda p)}$ is convex in p. By induction the proof is complete. □

An algorithm for computing an approximation, denoted $R_{S,u}$, to $r_{S,u}$ is as follows: start from u, recurse to some depth ℓ using Remark 2.2, replacing r's with R's. At the bottom of the recursion, set the relevant approximations $R_{\tilde{S},v} = p_v$. We now want to show that these approximations $R_{S,u}$ get close to $r_{S,u}$.

Note 2.4 *Ideally, we want the approximations R to improve accuracy as we move up the recursion tree starting from the bottom. Unfortunately, this is not exactly true, because the p_v values attached to each vertex can influence the error. The trick is in finding a good measure of how the errors propagate up the recursion tree, and properly bounding that measure.*

3 Error Analysis of the Cut-Off Recursion

Consider 1 step of recursion, from u down to its $v_1, \ldots, v_d$ neighbors. Using the exact values $r_{S,u}$ or approximations $R_{S,u}$ doesn't matter in what follows; the recursive relationship is the same. Taking a total differential of the 1-step recursive expression, we have

$$dr_{S,u} = d\left(p_u \cdot \prod_{i=1}^{d} \frac{1}{1 - r_{S_i,v_i}}\right) = \sum_{i=1}^{d} \frac{\partial r_{S,u}}{\partial r_{S_i,v_i}} dr_{S_i,v_i} = \sum_{i=1}^{d} r_{S,i} \cdot \frac{1}{1 - r_{S_i,v_i}} dr_{S_i,v_i}.$$

This expression relates how infinitesimal changes in the r_{S_i,v_i} values influence the change in $r_{S,u}$. It turns out that this is closely related to the scaling of p within Shearer's region. If we consider

$r_{S,u}((1+t)p)$ as a function of t, and take a derivative with respect to t, we obtain

$$\begin{aligned}\frac{d}{dt}(r_{S,u}((1+t)p)) &= \frac{d}{dt}(1+t)p_u \prod_{i=1}^{d} \frac{1}{1-r_{S_i,v_i}((1+t)p)} \\ &= p_u \prod_{i=1}^{d} \frac{1}{1-r_{S_i,v_i}((1+t)p)} \\ &\quad + \sum_{j=1}^{d} (1+t)p_u \left(\prod_{i=1}^{d} \frac{1}{1-r_{S_i,v_i}((1+t)p)}\right) \frac{1}{1-r_{S_i,v_i}((1+t)p)} \frac{dr_{S_i,v_i}((1+t)p)}{dt}.\end{aligned}$$

Evaluating this derivative at $t=0$ yields

$$\begin{aligned}\frac{d}{dt}(r_{S,u}((1+t)p))\Big|_{t=0} &= r_{S,u}(p) + \sum_{j=1}^{d} r_{S,u}(p) \frac{1}{1-r_{S_i,v_i}(p)} \frac{dr_{S_i,v_i}((1+t)p)}{dt}\Big|_{t=0} \\ &= r_{S,u}(p)\left(1 + \sum_{j=1}^{d} \frac{1}{1-r_{S_i,v_i}(p)} \frac{dr_{S_i,v_i}((1+t)p)}{dt}\Big|_{t=0}\right).\end{aligned}$$

Definition 3.1 *For a set of vertices V and vertex v, let*

$$\beta_{V,v} := \frac{dr_{V,v}((1+t)p)}{dt}\Big|_{t=0}.$$

Intuitively, β is a measure of how errors propagate upward in an infinitesimal deviation from the exact r values. Our deviations are not infinitesimal, but we will show that the error propagation from infinitesimal deviation is actually the worst (as you deviate more from r, the additional error propagates less).

The above derivation proved the following.

Lemma 3.2 $\beta_{S,u} = r_{S,u}\left(1 + \sum_{j=1}^{d} \frac{\beta_{S_j,v_j}}{1-r_{S_j,v_j}}\right)$.

The issue of a finite deviation from r vs. an infinitesimal deviation from r is resolved in the following lemma.

Lemma 3.3 *If $R_{S_i,v_i} \in [r_{S_i,v_i} - \delta\beta_{S_i,v_i}, r_{S_i,v_i}]$, then $R_{S,u} \in [r_{S,u} - \delta\beta_{S,u}, r_{S,u}]$.*

Proof: Note that

$$\frac{\partial r_{S,u}}{\partial r_{S_i,v_i}} = \frac{r_{S,u}}{1-r_{S_i,v_i}}$$

is an increasing function of p, since $r_{S,u}(p)$ are increasing functions of p and $1/(1-x)$ is increasing on $x \in [0,1)$. Then

$$\begin{aligned} R_{S,u}(p) = p_u \prod_{i=1}^{d} \frac{1}{1 - R_{S_i,v_i}} &\geq p_u \prod_{i=1}^{d} \frac{1}{1 - (r_{S_i,v_i} - \delta\beta_{S_i,v_i})} \\ &\geq p_u \prod_{i=1}^{d} \frac{1}{1 - r_{S_i,v_i}} - \sum_{i=1}^{d} \delta\beta_{S_i,v_i} \frac{\partial r_{S,u}}{\partial r_{S_i,v_i}} \\ &= r_{S,u} - \sum_{i=1}^{d} \delta\beta_{S_i,v_i} \frac{r_{S,u}}{1 - r_{S_i,v_i}} \\ &= r_{S,u} - \delta r_{S,u} \sum_{i=1}^{d} \frac{\beta_{S_i,v_i}}{1 - r_{S_i,v_i}} \\ &\geq r_{S,u} - \delta\beta_{S,u} \end{aligned}$$

where the first inequality is by hypothesis, the second by convexity of $1/(1-x)$ and r's, and the last inequality is by the previous lemma. The inequality $R_{S,u} \leq r_{S,u}$ follows from

$$R_{S,u} = p_u \prod_{i=1}^{d} \frac{1}{1 - R_{S_i,v_i}} \leq p_u \prod_{i=1}^{d} \frac{1}{1 - r_{S_i,v_i}}$$

by hypothesis on R_{S_i,v_i}'s and $1/(1-x)$ increasing. □

Remark 3.4 *It is possible to obtain a better analysis by taking advantage of the "1+" part from the recurrence for $\beta_{S,u}$. This improves the quantitative error estimation but requires more work.*

Now we would like to ensure that the β error propagation measures don't blow up, and furthermore that they do decrease (shrink the lower bound on R as we move up the recursive tree).

Lemma 3.5 *Assume ε is small enough so that $(1+\varepsilon)p$ is in Shearer's region.*

1. $\beta_{S,u} \leq \frac{1-r_{S,u}}{\varepsilon}$
2. $r_{S,u} \leq \beta_{S,u} \leq \left(1 + \frac{d}{\varepsilon}\right) r_{S,u}$
3. $r_{S,u} \leq \frac{1}{1+\varepsilon}$.

Proof:

1. $r_{S,u}((1+\varepsilon)p) \geq r_{S,u}(p) + \varepsilon \cdot \left.\frac{dr_{S,u}((1+t)p)}{dt}\right|_{t=0}$ by convexity of $r_{S,u}$, and $r_{S,u} < 1$. Noting that $\left.\frac{dr_{S,u}((1+t)p)}{dt}\right|_{t=0} = \beta_{S,u}$, we obtain $1 > r_{S,u}(p) + \varepsilon\beta_{S,u}$, and rearranging gives 1.

2. From Lemma 3.2, clearly $r_{S,u} \leq \beta_{S,u} \leq r_{S,u}\left(1 + \sum_{i=1}^{d} \frac{\beta_{S_j,v_j}}{1-r_{S_j,v_j}}\right)$. Applying 1 to all the $\beta_{S_j,v_j}/(1 - r_{S_j,v_j})$ in the sum gives 2.

3. Finally, $r_{S,u} \leq \beta_{S,u} \leq \frac{1-r_{S,u}}{\varepsilon}$ from 1 and 2. Solving for $r_{S,u}$ we get $r_{S,u} \leq \frac{1}{1-\varepsilon}$.

□

This result tells us that the errors don't propagate too much. But we want more than this; we want the degree to which they propagate to go down as we move up the recursion tree. To show this, we employ a dirty trick similar to last lecture.

Assume that ε is small enough so that $(1+\varepsilon)^2 p$ is in Shearer's region, in order to have $(1+\varepsilon)p$ still have slack $(1+\varepsilon)$ to stay in Shearer's region. Imagine we are trying to compute at p but are doing error analysis with respect to $\beta_{S,u}((1+\varepsilon)p) =: \beta'_{S,u}$. The above lemma gave us

$$\beta'_{S,u} \leq \left(1+\frac{d}{\varepsilon}\right) r'_{S,u} \leq \left(1+\frac{d}{\varepsilon}\right)\left(\frac{1}{1+\varepsilon}\right),$$

where $r'_{S,u} := r_{S,u}((1+\varepsilon)p)$. We also know that

$$r'_{S,u} \geq (1+\varepsilon) r_{S,u} \tag{3.1}$$

from the convexity of the $r_{S,u}$. The actual errors of $r_{S,u}$ propagate according to

$$\beta_{S,u} = r_{S,u}\left(1+\sum_{i=1}^{d}\frac{\beta_{S_i,v_i}}{1-r_{S_i,v_i}}\right)$$

at worst. However, the recurrence for $\beta'_{S,u}$ has an additional $(1+\epsilon)$ factor due to $r'_{S,u} \geq (1+\epsilon)r_{S,u}$.Therefore, since the errors at the bottom of the recursion are $r_{T_\ell,\ell} - p_\ell \leq r_{T_\ell,\ell} \leq \beta'_{T_\ell,\ell}$. Then at the root of the recursive tree, u^*, the error is at most

$$\frac{\beta'_{S,u^*}}{(1+\varepsilon)^\ell}.$$

Set the depth of recursion $\ell := \frac{1}{\varepsilon}\log\left(\frac{1+d/\varepsilon}{\varepsilon d}\right)$. This makes

$$\begin{aligned}
|R_{S,u^*} - r_{S,u^*}| &\leq \frac{\beta'_{S,u^*}}{(1+\varepsilon)^\ell} \\
&\leq (1+d/\varepsilon)\, r'_{S,u^*} \\
&\leq (1+d/\varepsilon)\, r'_{S,u^*} e^{-\log(1+d/\varepsilon)+\log(\varepsilon\delta)} \\
&= r'_{S,u^*}\varepsilon\delta \\
&\leq (1-r'_{S,u^*})\delta \\
&\leq (1-r_{S,u^*})\delta
\end{aligned}$$

where the second to last inequality follows from 3 of Lemma 3.5. Our goal was to compute $\breve{q}_V = \prod_{i=1}^{n}(1-r_{V_i,i})$. Setting $\delta = \delta'/n$, we calculate the recursive running time to be

$$O\left(nd^\ell\right) = O\left(nd^{\frac{1}{\varepsilon}\log\left(\frac{nd}{\delta'\varepsilon^2}\right)}\right) = O\left(n\left(\frac{nd}{\delta'\varepsilon^2}\right)^{\frac{1}{\varepsilon}\log d}\right).$$

This is quasi-polynomial for constant ε slack in the Shearer region (the constant ε is important; otherwise this blows up). For constant d, this is polynomial.

Remark 3.6 *If we had used the "1+" in the recurrence for $\beta_{S,u}$, with a more careful analysis, this could be improved to $1/\sqrt{\varepsilon}$ in the exponent.*