Stanford University	**Spring 2018**
Math 233: Non-constructive methods in combinatorics	**Instructor: Jan Vondrák**
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Lecture 12. The Heilmann-Lieb Theorem

Recall that earlier in the course, we appealed to the matching polynomial and an upper bound on its (real) roots. Let us remind ourselves of the definition.

Definition 12.1 *For a graph $G = (V, E)$ with edge weights w_e, the matching defect polynomial is*

$$\mathcal{M}_G(x) = \sum_{\text{matching } M} (-1)^{|M|} x^{n-2|M|} \prod_{e \in M} w_e.$$

Theorem 12.2 (The Heilmann-Leib Theorem) *The matching defect polynomial for a graph G with edge weights $w_{uv} > 0$ satisfies:*

- *$\mathcal{M}_G$ is real-rooted.*
- *For a connected graph G with $|V(G)| \geq 3$: Let $W_u = \sum_{v \in \Gamma(u)} w_{uv} - \min_{u' \in \Gamma(u)} w_{uu'}$ and $B = \max_{u \in V} W_u$. Then the maximum root of $\mathcal{M}_G$ is at most $2\sqrt{B}$.*

Note that the result does not hold when G is a single edge, since then $\mathcal{M}_G(x) = x^2 - w$ has a positive root while $W_u = W_v = 0$. Since the matching polynomial of a disconnected graph is a product of the matching polynomials of the connected components, the theorem also holds for graphs G that do not contain isolated edges.

First, we prove the first bullet point for G complete, $|V(G)| \geq 1$, and $w_{uv} > 0 \; \forall u \neq v$.

Proof: We have the following recursion: $\mathcal{M}_\emptyset(x) = 1$, and

$$\mathcal{M}_G(x) = x\mathcal{M}_{G \setminus \{u\}}(x) - \sum_{v \in V \setminus \{u\}} w_{uv} \mathcal{M}_{G \setminus \{u,v\}}(x).$$

(The first term corresponds to matchings avoiding u and the second corresponds to matchings covering u.)

We prove by induction on $|V(G)|$ that: (*)
$\mathcal{M}_G$ is real-rooted, with distinct simple roots and for all $u \in V$, $\mathcal{M}_{G \setminus \{u\}}$ strictly interlaces $\mathcal{M}_G$.

For the base case: $G = \{u\}$, $\mathcal{M}_G(x) = x$ and $\mathcal{M}_{G \setminus \{u\}}(x) = \mathcal{M}_\emptyset(x) = 1$ so (*) holds.

Consider $|V(G)| = n$ assume that (*) holds for $|V(G')| \leq n - 1$. Let $\lambda_{n-1} < \lambda_{n-2} < ... < \lambda_1$ be the roots of $\mathcal{M}_{G \setminus \{u\}}$ (real and distinct by the inductive hypothesis). We have

$$\mathcal{M}_G(\lambda_i) = \lambda_i \mathcal{M}_{G \setminus \{u\}}(\lambda_i) - \sum_{v \neq u} w_{uv} \mathcal{M}_{G \setminus \{u,v\}}(\lambda_i) = -\sum_{v \neq u} w_{uv} \mathcal{M}_{G \setminus \{u,v\}}(\lambda_i). \tag{12.1}$$

By the inductive hypotesis, $\mathcal{M}_{G\setminus\{u,v\}}$ strictly interlaces $\mathcal{M}_{G\setminus\{u\}}$. This means that $\mathcal{M}_{G\setminus\{u,v\}}(\lambda_i)$ alternates signs for $i = 1, 2, 3, \ldots$ (as between two consecutive values of λ_i there is exactly one simple root of $\mathcal{M}_{G\setminus\{u,v\}}$). Moreover, there is no root of $\mathcal{M}_{G\setminus\{u,v\}}$ greater than λ_1 and $\mathcal{M}_{G\setminus\{u,v\}}$ has a positive highest coefficient, so $\mathcal{M}_{G\setminus\{u,v\}}(\lambda_1) > 0$. This implies that $(-1)^{i-1}\mathcal{M}_{G\setminus\{u,v\}}(\lambda_i) > 0$.

By (12.1), as $w_{uv} > 0\ \forall u \neq v$, we have $(-1)^i\mathcal{M}_G(\lambda_i) > 0$. Thus, $\mathcal{M}_G$ has a root in each interval $(\lambda_{i-1}, \lambda_i)$ for $i = 1, 2, ..., n$. Moreover, $\mathcal{M}_G(\lambda_1) < 0$ and $\lim_{x\to\infty}\mathcal{M}_G(x) = \infty$ so $\mathcal{M}_G$ has a root larger than λ_1.

Similarly $(-1)^{n-1}\mathcal{M}_G(\lambda_{n-1}) > 0$ while $\lim_{x\to-\infty}\mathcal{M}_G(x) = (-1)^n\infty$ since $\deg \mathcal{M}_G = n$, we see that there is a root of $\mathcal{M}_G$ below λ_{n-1}. Hence, as $\mathcal{M}_G$ has at most n roots, it has exactly n roots interlacing $\{\lambda_{n-1}, ..., \lambda_1\}$. □

Note that non-edges in G can be thought of as zero weights $w_{uv} = 0$. Hence it is enough to consider the complete graph with weights $w_{uv} \geq 0$. Our approach is to take a limit of graphs with $w'_{uv} > 0$ and apply the above result. We use the following general fact.

Lemma 12.3 *For a sequence of polynomials $f_1(z), f_2(z), ...$ in complex variable z, suppose that the degrees of f_m are uniformly bounded, $\Omega \subset \mathbb{C}$ is an open set, and $f_m \to f \in \mathbb{C}[z]$ coefficient-wise.*

If f_m has no roots in Ω for all m then either f has no root in Ω or $f \equiv 0$.

Proof: Assume that $f \not\equiv 0$ but f has a root $z_0 \in \Omega$. Choose $\rho > 0$ so that

$$\overline{B}_\rho(z_0) = \{z \in \mathbb{C} : |z - z_0| \leq \rho\} \subset \Omega$$

and there is no other root of f in $\overline{B}_\rho(z_0)$. (This can be done since f has finitely many roots.)

Since $f_m \to f$ pointwise and $\overline{B}_\rho(z_0)$ is compact, $f_m \to f$ uniformly on $\overline{B}_\rho(z_0)$.

Let $c = \min_{z\in\partial\overline{B}_\rho(z_0)} |f(z)|$. We have $c > 0$ because f has no roots in $\overline{B}_\rho(z_0)$. Take n_0 such that for all $n \geq n_0$, and for all $z \in \partial\overline{B}_\rho(z_0)$, we have $|f_m(z)| \geq \frac{c}{2}$. This can be done since $f_m \to f$ uniformly on $\partial\overline{B}_\rho(z_0)$. Since $f_m \to f$ coefficient-wise, we also have $f'_m \to f'$ coefficient-wise (note that the degree of the polynomials is uniformly bounded). Hence, $f'_m \to f'$ uniformly on $\partial\overline{B}_\rho(z_0)$.

As $|f_m(z)| \geq \frac{c}{2} > 0$ for $z \in \partial\overline{B}_\rho(z_0)$, we have $\frac{f'_m(z)}{f_m(z)} \to \frac{f'(z)}{f(z)}$ uniformly. Hence,

$$\int_{\partial\overline{B}_\rho(z_0)} \frac{f'_m(z)}{f_m(z)}dz \to \int_{\partial\overline{B}_\rho(z_0)} \frac{f'(z)}{f(z)}dz. \tag{12.2}$$

Let $f(z) = a\prod_{i=1}^n(z - \lambda_i)$. Then $\frac{f'(z)}{f(z)} = \sum_{i=1}^n \frac{1}{z-\lambda_i}$. Hence, $\int_{\partial\overline{B}_\rho(z_0)} \frac{f'(z)}{f(z)}dz = 2m\pi i$ where m is the multiplicity of root z_0 which is nonzero. On the other hand, for every n, we have $\int_{\partial\overline{B}_\rho(z_0)} \frac{f'_m(z)}{f_m(z)}dz = 0$ since there is no root of f_m in $\overline{B}_\rho(z_0) \subset \Omega$ so $\frac{f'_m(z)}{f_m(z)}$ is holomorphic in a neighborhood of $\overline{B}_\rho(z_0)$. This contradicts (12.2). □

Now let us finish the proof of Theorem 12.2.

Proof: Given $w_{uv} \geq 0$, we can take graphs $G^{(m)}$ with weights $w_{uv}^{(m)} = w_{uv}$ if $w_{uv} > 0$ and $w_{uv}^{(m)} = \frac{1}{m}$ if $w_{uv} = 0$. The sequence $\mathcal{M}_{G^{(m)}}(x)$ satisfies the conditions in the theorem above for $\Omega = \mathbb{C}\setminus\mathbb{R}$, by the proof above for strictly positive weights. By Lemma 12.3, $\mathcal{M}_G(x)$ has no zeros in $\mathbb{C}\setminus\mathbb{R}$ (since it is not identically 0) which means it is a real-rooted polynomial.

Now to the second bullet point. Let us assume now that G is not necessarily complete and all edges have positive weights. We prove by induction on $|V(G')|$ the following claim (with no assumption on connectedness and size of G):

For every proper induced subgraph $G' \subset G$, and $u \in V(G')$ which has a neighbor in $V(G) \backslash V(G')$, for all $x > 2\sqrt{B}$, $\mathcal{M}_{G'}(x) > 0$ and $\frac{\mathcal{M}_{G'}(x)}{\mathcal{M}_{G' \backslash \{u\}}(x)} \geq \sqrt{B}$. (**)

For the base case, $G' = \{u\}$. We have $\mathcal{M}_{G'}(x) = x, \mathcal{M}_{\emptyset}(x) = 1$ and $\frac{\mathcal{M}_{G'}(x)}{\mathcal{M}_{\emptyset}(x)} = x \geq \sqrt{B}$ for all $x > 2\sqrt{B}$.

For the inductive step, assume that $|V(G')| = k$, $u \in V(G')$ has a neighbor z outside of G', and (**) is true for all subgraphs of at most $k-1$ vertices. We have

$$\mathcal{M}_{G'}(x) = x\mathcal{M}_{G' \backslash \{u\}}(x) - \sum_{v \in \Gamma(u) \cap G'} w_{uv} \mathcal{M}_{G' \backslash \{u,v\}}(x).$$

Hence,

$$\frac{\mathcal{M}_{G'}(x)}{\mathcal{M}_{G' \backslash \{u\}}(x)} = x - \sum_{v \in \Gamma(u) \cap G'} w_{uv} \frac{\mathcal{M}_{G' \backslash \{u,v\}}(x)}{\mathcal{M}_{G' \backslash \{u\}}(x)}.$$

We apply the inductive hypothesis to $G' \setminus u$ and each $v \in \Gamma(u) \cap G'$. We have $\frac{\mathcal{M}_{G' \backslash \{u\}}(x)}{\mathcal{M}_{G' \backslash \{u,v\}}(x)} \geq \sqrt{B}$ for all $x \geq 2\sqrt{B}$. Hence,

$$\frac{\mathcal{M}_{G'}(x)}{\mathcal{M}_{G' \backslash \{u\}}(x)} = x - \sum_{v \in \Gamma(u) \cap G'} w_{uv} \frac{\mathcal{M}_{G' \backslash \{u,v\}}(x)}{\mathcal{M}_{G' \backslash \{u\}}(x)}$$

$$\geq x - \sum_{v \in \Gamma(u) \cap G'} w_{uv} \cdot \frac{1}{\sqrt{B}} \geq x - \left(\sum_{v \in \Gamma(u) \cap G} w_{uv} - w_{uz} \right) \cdot \frac{1}{\sqrt{B}}$$

$$\geq x - \frac{W_u}{\sqrt{B}} \geq x - \frac{B}{\sqrt{B}} \geq 2\sqrt{B} - \sqrt{B} = \sqrt{B}$$

where z is a neighbor of u in $G \backslash G'$. Hence we also have $\mathcal{M}_{G'}(x) > 0$.

To finish the proof, for G connected with at least 3 vertices, pick a vertex u with $\deg u \geq 2$. By the above, $\mathcal{M}_{G \backslash u}(x) > 0$ for $x > 2\sqrt{B}$. Also, we have $B \geq W_u = \sum_{v \in \Gamma(u)} w_{uv} - \min w_{uv} \geq \frac{1}{2} \sum_{v \in \Gamma(u)} w_{uv}$. Hence $\frac{\mathcal{M}_G(x)}{\mathcal{M}_{G \backslash \{u\}}(x)} \geq x - \sum_{v \in \Gamma(u)} w_{uv} \cdot \frac{1}{\sqrt{B}} > 2\sqrt{B} - \frac{2B}{\sqrt{B}} = 0$ and $\mathcal{M}_G(x) > 0$. □