Stanford University **Winter 2018**
Math 233: Non-constructive methods in combinatorics **Instructor: Jan Vondrák**
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Lecture 13. Stable polynomials and stability-preserving transformations

13.1 Stable polynomials

Stable polynomials generalize the notion of real-rootedness to complex variables. The following definition seems non-obvious at first, but turns out to be the right notion.

Definition 13.1 *$f \in \mathbb{C}[z_1, ..., z_n]$ is stable if $f \equiv 0$, or f has no root in $\mathcal{H}^n$ where $\mathcal{H} = \{z \in \mathbb{C} : Im(z) > 0\}$. A stable polynomial with real coefficients is called real stable.*

Lemma 13.2 *If f is a univariate real stable polynomial then f is real-rooted.*

Proof: Let $f(z) = \sum_{k=0}^{d} a_k z^k$ where $a_k \in \mathbb{R}$. Then the roots of f are either real or appear in conjugate pairs $\lambda, \overline{\lambda}$. Hence, $f(z) \neq 0 \ \forall z \in \mathcal{H} \Leftrightarrow f(z) \neq 0 \ \forall z \in -\mathcal{H}$. Thus, f must be real-rooted. □

Note that this is not true for more than one variable, for example: $f(z_1, z_2) = 1 - z_1 z_2$ is stable but not real-rooted (as its roots are all pairs $(z_1, z_2) \in \mathbb{C}^2$ such that $z_2 = 1/z_1$, which can be non-real but not both in $\mathcal{H}$).

Lemma 13.3 *$f \in \mathbb{C}[z_1, ..., z_n]$ has no roots in $\mathcal{H}^n$ iff for all $x \in \mathbb{R}^n, d \in \mathbb{R}_+^n$, $g(t) = f(x + td)$ (in variable $t \in \mathbb{C}$) has no zeros in $\mathcal{H}$.*

(Note: $\mathbb{R}_+ = \{x \in \mathbb{R} : x > 0\}$, i.e. all coordinates of d should be strictly positive.)

Proof: If f has no roots in $\mathcal{H}^n$, $f(z_1, ..., z_n) = 0$ for some $z_1, ..., z_n \in \mathcal{H}$. Then take $x_i = Re(z_i), d_i = Im(z_i)$; we have $g(i) = f(x + id) = 0$ so g has a zero in $\mathcal{H}$.

If $g(t) = 0$ for $t \in \mathcal{H}$ then $f(x_1 + td_1, ..., x_n + td_n) = 0$ and $x_i + td_i \in \mathcal{H}$ since $Im(x_i + td_i) = d_i \cdot Im(t) > 0$ since $t \in \mathcal{H}$ and $d \in \mathbb{R}_+^n$. Hence, f is not stable. □

Note: Because we consider the zero polynomial stable, it is not correct to say that $f(z)$ is stable iff every $g(t) = f(x + td)$ as above is stable. A counterexample is $f(z_1, z_2) = z_1 - z_2$ which is not stable (having a root (i, i)) but $g(t) = f(x_1 + td_1, x_2 + td_2) = (x_1 - x_2) + t(d_1 - d_2)$ is either real-rooted (linear) or constant, hence stable.

Next we prove that stability is closed under taking limits. This is a generalization of a property of univariate polynomials that we already proved in the last lecture.

Lemma 13.4 *Let $f_1, f_2, \dots \in \mathbb{C}[z_1, \dots, z_n]$ be a sequence of polynomials of bounded degree with no zeros in $\mathcal{H}^n$ and assume that $f_m \to f$ coefficient-wise. Then either $f \equiv 0$ or f has no zeros in $\mathcal{H}^n$.*

Proof: We will prove this by induction on n. We proved the case of $n = 1$ in the previous lecture (noting that $\mathcal{H}$ is open).

Now assume that $a \in \mathcal{H}$ is fixed. Then $f_m(z_1, \ldots, z_{n-1}, a) \to f(z_1, \ldots, z_{n-1}, a)$ and by the induction hypothesis it must be that either

1. $f(z_1, \ldots, z_{n-1}, a) \equiv 0$ or
2. $f(z_1, \ldots, z_{n-1}, a)$ has no roots in $\mathcal{H}^{n-1}$.

If 1. happens for all $a \in \mathcal{H}$, then $f \equiv 0$ and we are done. Similarly if 2. happens for all $a \in \mathcal{H}$, then f has no zeros in $\mathcal{H}^n$ and we are done.

But what if 1. happens for some (but not all) $a \in \mathcal{H}$? Look at $g(z) = f(i, i, \ldots, i, z) \in \mathbb{C}[z]$. Since $g(z) = f(i, \ldots, i, z) = \lim_{m\to\infty} f_m(i, \ldots, i, z)$, we have $g(z) = 0$ for some (but not all) $z \in \mathcal{H}$. This means that $g \not\equiv 0$, but g has zeros in $\mathcal{H}$. This is in contradiction with continuity of the roots of univariate polynomials, as none of the $f_m(i, \ldots, i, z)$ have a root in $\mathcal{H}$. $\square$

13.2 Operators preserving real stability

Let us now discuss a list of basic operations that preserve stability.

Theorem 13.5 *The following transformations on $\mathbb{C}[z_1, \ldots, z_n]$ preserve stability.*

1. ***Permutation**: $f(z_1, \ldots, z_n) \mapsto f(z_{\sigma(1)}, \ldots, z_{\sigma(n)})$ for some $\sigma \in S_n$.*
2. ***Scaling**: $f(z_1, \ldots, z_n) \mapsto f(az_1, \ldots, z_n)$ for $a \in \mathbb{R}_+$.*
3. ***Diagonalization**: $f(z_1, z_2, \ldots, z_n) \mapsto f(z_2, z_2, z_3, \ldots, z_n)$.*
4. ***Inversion**: $f(z_1, \ldots, z_n) \mapsto z_1^d f(-\frac{1}{z_1}, \ldots, z_n)$ where $d = \deg_{z_1} f$.*
5. ***Specialization**: $f(z_1, \ldots, z_n) \mapsto f(a, z_2, \ldots, z_n)$ for $a \in \mathcal{H} \cup \mathbb{R}$.*
6. ***Differentiation**: $f(z_1, \ldots, z_n) \mapsto \frac{\partial}{\partial z_1} f(z_1, \ldots, z_n)$.*

Proof: Let us go through the operations, one by one.

Permutation: Permutation of coordinates is a bijection between $\mathcal{H}^n$ and itself.

Scaling: Scaling by a constant $a \in \mathbb{R}_+$ is a bijection between $\mathcal{H}$ and itself.

Diagonalization: By definition $f(z_2, z_2, \ldots, z_n) \neq 0$ for $z_2, \ldots, z_n \in \mathcal{H}$.

Inversion: The map $z_1 \mapsto -\frac{1}{z_1}$ is a bijection between $\mathcal{H}$ and itself.

Specialization: Immediate for $a \in \mathcal{H}$, as by definition $f(a, z_2, \ldots, z_n) \neq 0$ for $a, z_2, \ldots, z_n \in \mathcal{H}$. For $a \in \mathbb{R}$, we use the limiting argument of Lemma 13.4.

Differentiation: This follows immediately from the Gauss-Lucas theorem which states that for $f \in \mathbb{C}[z]$, the roots of f' lie in the convex hull of the roots of f (note that the complement of $\mathcal{H}$ is convex). Next we will see a proof of the Gauss-Lucas theorem. $\square$

Theorem 13.6 (Gauss-Lucas) *If $f \in \mathbb{C}[z]$ (single complex variable), then the roots of f' are in the convex hull of the roots of f.*

Proof: Let $\lambda_1, \dots, \lambda_n$ be the roots of f and assume that $f'(z) = 0$ for some z. If $f(z) = 0$, we are done, so assume that $f(z) \neq 0$. Therefore

$$0 = \frac{f'(z)}{f(z)} = \sum_i \frac{1}{z - \lambda_i} = \sum_i \frac{\overline{z - \lambda_i}}{|z - \lambda_i|^2}.$$

We can rewrite this as

$$\sum_i \frac{z}{|z - \lambda_i|^2} = \sum_i \frac{\lambda_i}{|z - \lambda_i|^2},$$

or equivalently

$$z = \left(\sum_i \frac{1}{|z - \lambda_i|^2} \right)^{-1} \sum_i \frac{\lambda_i}{|z - \lambda_i|^2},$$

which shows that z is a convex combination of $\lambda_1, \dots, \lambda_n$. □

13.3 Stability from PSD matrices

A natural and interesting class of stable polynomials can be derived from PSD matrices. Recall the following definitions about Hermitian and PSD matrices.

Definition 13.7 *For a matrix $A \in \mathbb{C}^{n \times n}$, define A^* to be the matrix where $(A^*)_{ij} = \overline{A_{ji}}$ (transpose and complex-conjugate).*

Definition 13.8 *A matrix $H \in \mathbb{C}^{n \times n}$ is Hermitian iff $H^* = H$.*

Definition 13.9 *A Hermitian matrix H is positive semidefinite (PSD, or $H \succeq 0$), iff for all $\mathbf{x} \in \mathbb{C}^n$ we have $\mathbf{x}^* H \mathbf{x} \geq 0$.*

Definition 13.10 *A Hermitian matrix H is positive definite (PD, or $H \succ 0$), iff for all $0 \neq \mathbf{x} \in \mathbb{C}^n$ we have $\mathbf{x}^* H \mathbf{x} > 0$.*

The following lemma describes one of the most important families of real stable polynomials.

Lemma 13.11 *Given PSD matrices $A_1, \dots, A_n$ and Hermitian B, the polynomial*

$$f(z_1, \dots, z_n) = \det(\sum_j z_j A_j + B)$$

is real stable.

Proof: We can assume w.l.o.g. that $A_1, \dots, A_n \succ 0$, since all PSD matrices are limits of PD matrices; real stability follows from taking limits and lemma 13.4.

We will use Lemma 13.3. Let $x \in \mathbb{R}^n$ and $d \in \mathbb{R}^n_+$ and substitute $z_j = x_j + t d_j$: we get

$$g(t) = \det(t \overbrace{\sum_j d_j A_j}^{M \succ 0} + \sum_j x_j A_j + B).$$

Note that the matrix $M = \sum_i d_i A_i$ is positive definite. We can therefore define the square root and inverse square roots for it: decompose M as U^*DU, where U is a unitary matrix and D is diagonal with positive elements. Then

$$M^{\frac{1}{2}} = U^* D^{\frac{1}{2}} U,$$
$$M^{-\frac{1}{2}} = U^* D^{-\frac{1}{2}} U.$$

Now we have

$$g(t) = \det(tM + \sum_j x_j A_j + B) = \det(M^{\frac{1}{2}}(tI + M^{-\frac{1}{2}}(\sum_j x_j A_j + B)M^{-\frac{1}{2}})M^{\frac{1}{2}})$$
$$= \det(M) \underbrace{\det(tI + \overbrace{M^{-\frac{1}{2}}(\sum_j x_j A_j + B)M^{-\frac{1}{2}}}^{\text{Hermitian}})}_{\text{characteristic polynomial}}.$$

Therefore g is a multiple of the characteristic polynomial of a Hermitian matrix, which means that it has real roots, i.e. no roots in $\mathcal{H}$. □

The class of polynomials in lemma 13.11 is usually the basis for deriving other real stable polynomials, by applying real stability preserving transformations that will be discussed shortly. However this class is rich enough that for univariate and bivariate polynomials, no other transformation is needed as witnessed by the following theorem.

Theorem 13.12 (Helton-Vinnikov) *A polynomial $p \in \mathbb{C}[z_1, z_2]$ is real stable iff p can be written as $k \cdot \det(z_1 A + z_2 B + C)$ where $k \in \mathbb{R}$ and A, B, C are real symmetric matrices where $A, B \succeq 0$.*

14 The spanning tree polynomial

Example 14.1 (Spanning tree polynomial) *For a graph $G = (V, E)$ define the spanning tree polynomial $\mathcal{T}_G \in \mathbb{C}[z_e : e \in E]$ as*

$$\mathcal{T}_G(z_e : e \in E) = \sum_{\substack{T \subseteq E \\ T \text{ is a spanning tree}}} \prod_{e \in T} z_e.$$

Lemma 14.2 *The polynomial $\mathcal{T}_G$ is real stable.*

Proof: We use the matrix-tree theorem. Let L_G be the Laplacian matrix of the graph G defined by

$$(L_G)_{ii} = \deg(i),$$
$$(L_G)_{ij} = \begin{cases} -1 & \text{if } (i,j) \in E, \\ 0 & \text{otherwise.} \end{cases}$$

For an edge $e = (i, j)$ let $b_e = \delta_i - \delta_j$ where δ_i is i-th element of the standard basis in $\mathbb{R}^n$. In other words let

$$b_e = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ \vdots \\ -1 \\ \vdots \\ 0 \\ 0 \end{pmatrix},$$

where the only nonzero entries are the i-th and j-th entries. It is easy to see that

$$L_G = \sum_{e \in E} b_e b_e^\top.$$

The matrix-tree theorem states that the number of spanning trees of G is equal to $\det(\widetilde{L_G})$ where $\widetilde{L_G}$ is obtained from L_G by removing the first row and column. If we let $\widetilde{b_e}$ be the vector obtained from b_e by removing the first component, then $\widetilde{L_G} = \sum_{e \in E} \widetilde{b_e}\widetilde{b_e}^\top$. More generally one can see that

$$\mathcal{T}_G(z_e : e \in E) = \det(\sum_{e \in E} z_e \underbrace{\widetilde{b_e}\widetilde{b_e}^\top}_{\succeq 0}).$$

From lemma 13.11, it follows that the polynomial $\mathcal{T}_G$ is real stable. □

We will prove the following combinatorial fact using the tools we have covered.

Lemma 14.3 *Let T be a uniformly random spanning in the graph $G = (V, E)$. Assume that $F \subseteq E$ is fixed. Then $|T \cap F|$ has the same distribution as the sum of some independent $\{0, 1\}$-valued random variables.*

Proof: Let $Q_F \in \mathbb{R}[z]$ be defined as

$$Q_F(z) = \mathbb{E}[z^{|T \cap F|}] = \sum_{k=0}^{|F|} q_k z^k,$$

where $q_k = \Pr[|T \cap F| = k]$.

But note that Q_F can be obtained from $\mathcal{T}_G$ in the following way:

$$Q_F(z) = \frac{1}{\text{number of spanning trees}} \mathcal{T}_G(\underbrace{z, \ldots, z}_{F}, 1, \ldots, 1).$$

Therefore Q_F is real stable. Since Q_F is univariate this means that Q_F has real roots. Since Q_F has positive coefficients, these roots must be negative. Any number in $\mathbb{R}_-$ can be written as $-\frac{p}{1-p}$ for some $p \in (0, 1)$. Therefore we can write

$$Q_F(z) = c \cdot \prod_i (z + \frac{p_i}{1 - p_i}),$$

or equivalently

$$Q_F(z) = k \cdot \prod_i ((1 - p_i)z + p_i),$$

for $k, c \in \mathbb{R}$ and $p_i \in (0, 1)$.

Note that $k = Q_F(1) = 1$. Therefore

$$Q_F(z) = \prod_i ((1 - p_i)z + p_i).$$

This is the generating polynomial of the sum of independent $\{0, 1\}$-valued random variables with the probability of being 0 given by p_i. $\square$