Stanford University — **Winter 2018**
Math 233A: Non-constructive methods in combinatorics — **Instructor: Jan Vondrák**
Lecture date: February 22, 2018 — **Scribe: Yuval Wigderson**

Lecture 14. Characterization of linear stability-preserving transformations

Our next task is to classify which linear operators are stability-preserving.

Theorem 14.1 (Borcëa-Brandon 2009) *Let $\mathbb{C}_k[z_1, \ldots, z_n]$ be the set of polynomials where each z_j has degree at most k, let $T : \mathbb{C}_k[z_1, \ldots, z_n] \to \mathbb{C}[z_1, \ldots, z_n]$ be a linear map, and suppose that $\dim(\mathrm{Im}(T)) \geq 2$. Define the* symbol *of T to be the polynomial*

$$G_T(z, w) = T\left((z_1 + w_1)^k (z_2 + w_2)^k \cdots (z_n + w_n)^k\right)$$

where T acts on the z_j variables, treating the w_j as constants. Then T is stability-preserving if and only if G_T is stable.

Example 14.2 *Recall the matching polynomial with positive edge weights $w_{uv} > 0$ for all $(u, v) \in E$. We will prove that it is stable by using this result.*

Define

$$Q(z_v : v \in V) = \prod_{(u,v) \in E} (1 - w_{uv} z_u z_v)$$

Then we claim that Q is stable. To check this, it suffices to check that each of the factors is stable. But if a factor vanishes, then $z_u z_v = 1/w_{uv} > 0$, and the product of two numbers in $\mathcal{H}$ can never be a positive real.

Now, consider the monomials in Q where each variable is at most linear; these monomials precisely correspond to matchings, since they ensure that we never get two edges that meet at a vertex, since such a vertex would have degree ≥ 2.

So consider the operator MAP (multi-affine part) which simply eliminates all monomials where any variable has degree ≥ 2. Formally,

$$MAP\left(\prod_{u \in S} z_u^{d_u}\right) = \begin{cases} \prod_{u \in S} z_u^{d_u} & \text{if } d_u \leq 1 \text{ for all } u \in S \\ 0 & \text{otherwise} \end{cases}$$

and extended linearly to $\mathbb{C}_k[z_1, \ldots, z_n]$, where k is the maximal degree in the graph. Its symbol is

$$
\begin{aligned}
G_{MAP} &= MAP\left(\prod_{u\in V}(z_u + w_u)^k\right) \\
&= MAP\left(\prod_{u\in V}(w_u^k + kw_u^{k+1}z_u + \cdots)\right) \\
&= \prod_{u\in V}(w_u^k + kw_u^{k-1}z_u) \\
&= \prod_{u\in V} w_u^{k-1}(w_u + kz_u)
\end{aligned}
$$

and each of these factors is non-zero for $w_u, z_u \in \mathcal{H}$. Thus, G_{MAP} is stable, so MAP is stability-preserving. Applying it to Q gives

$$MAP[Q](z) = \sum_{\text{matchings } M} (-1)^{|M|} \prod_{(u,v)\in M} w_{uv} \prod_{(u,v)\in M} (z_u z_v)$$

which is just the matching polynomial when evaluated with all z_u equal to some constant z (up to some renormalization). Thus, the matching polynomial is stable.

Proof:[Partial proof of Thoerem 14.1] We will prove the forward implication in the case where $k = 1$. We then have

$$\prod_{j=1}^{n}(z_j + w_j) = \sum_{S\subseteq[n]} z^S w^{\overline{S}}$$

Therefore,

$$G_T(z, w) = T\left(\prod_{j=1}^{n}(z_j + w_j)\right) = \sum_{S\subseteq[n]} T(z^S)w^{\overline{S}}$$

Our goal is to prove that if G_T is stable, then T is stability-preserving on $\mathbb{C}_1[z_1, \ldots, z_n]$.

Lemma 14.3 (Lieb-Sokal) *Suppose that $f(z_1, \ldots, z_n) + wg(z_1, \ldots, z_n)$ is a stable polynomial in in $\mathbb{C}[z_1, \ldots, z_n, w]$, and the degree of z_1 in g is at most 1. Then*

$$f(z_1, \ldots, z_n) - \frac{\partial g}{\partial z_1}(z_1, \ldots, z_n)$$

is stable in $\mathbb{C}[z_1, \ldots, z_n]$.

Suppose for example that $n = 1$ and $g(z) = -if(z)$. Thus, this says that if $(1 - iw)f(z)$ is stable, then $(1 + i\frac{\partial}{\partial z})f$ is also stable. However, $1 - iw$ is stable, so the assumption is equivalent to $f(z)$ being stable. So for example, if $f(z) = z$, then the conclusion is that $z + i$ is stable, which is true. This demonstrates that the sign-flip in the statement is important, since if we didn't flip the sign, we'd get that $z - i$ is stable, which is false.

Proof: From the assumption that $f + wg$ is stable, we get that f and g are stable, since they can be obtained from $f + wg$ by stability-preserving transformations (namely plugging in $w = 0$ and differentiating in w, respectively). Define

$$h(z, w) = wg\left(z_1 - \frac{1}{w}, z_2, \ldots, z_n\right)$$

which is a polynomial since we assumed z_1 had degree at most 1 in g. It's a stable polynomial, since if $w \in \mathcal{H}$, then so is $-1/w$, so any root in $\mathcal{H}^{n+1}$ of h would give a root in $\mathcal{H}^n$ of g, contradicting its stability. We can equivalently write

$$h(z, w) = wg(z) - \frac{\partial g}{\partial z_1}$$

because g is affine in z_1. That implies that any monomial m containing z_1 will become $wm - 1/z_1$, and all monomials not containing z_1 will just be multiplied by w, which implies the above formula. Therefore,

$$\operatorname{Im}\left(\frac{\frac{\partial g}{\partial z_1}}{g(z)}\right) \leq 0$$

for all $z \in \mathcal{H}^n$. Indeed, this just says that there is no way of picking $w \in \mathcal{H}$ to make $wg(z) - \frac{\partial g}{\partial z_1} = 0$, which is precisely the stability of h.

By the same argument, from the fact that $f(z) + wg(z)$ is stable, we see that for all $z \in \mathcal{H}^n$, $\operatorname{Im}(f(z)/g(z)) \geq 0$. Putting these together, we get that for all $z \in \mathcal{H}^n$,

$$\operatorname{Im}\left(\frac{f(z) - \frac{\partial g}{\partial z_1}}{g(z)}\right) \geq 0$$

That implies that $f(z) - \partial g/\partial z_1 + wg(z)$ must be stable. Plugging in $w = 0$ gives us the result we wanted. □

Returning to the proof, suppose that the symbol

$$G_T(z, w) = \sum_{S \subseteq [n]} T(z^S) w^{\overline{S}}$$

is stable. Fix some other multi-affine polynomial

$$f(z) = \sum_{S \subseteq [n]} a_S z^S$$

and assume that it's stable; our goal is to show that $T[f]$ is also stable. Consider

$$w_1 \cdots w_n G_T\left(z_1 - \frac{1}{w_1}, \ldots, z_n - \frac{1}{w_n}\right) = \sum_{S \subseteq [n]} (-1)^{|\overline{S}|} T(z^S) w^S$$

which is still stable. Multiplying by $f(v_1, \ldots, v_n)$ preserves stability, and gives the polynomial

$$\sum_{S \subseteq [n]} (-1)^{|\overline{S}|} T(z^S) w^S \sum_{R \subseteq [n]} a_R v^R.$$

Repeatedly applying the Lieb-Sokal lemma, which allows us to replace w_j by $-\partial/\partial v_j$, yields the stable polynomial

$$\sum_{S\subseteq[n]}(-1)^{|\overline{S}|+|S|}T(z^S)\frac{\partial^S}{\partial v^S}\sum_{R\subseteq[n]}a_Rv^R.$$

We now plug in 0 for all v_j, killing any term were $S \neq R$ and maintaining stability. So we get the stable polynomial

$$(-1)^n\sum_{S\subseteq[n]}T(z^S)a_S$$

which is $\pm T[f]$. Thus, T is stability-preserving.

In order to get the characterization for general polynomials (not just multi-affine) ones, one uses a technique called "polarization": we replace higher-order terms z_j^k by the product of clone variables $z_{j1}z_{j2}\cdots z_{jk}$. If we have $k' < k$, then $z_j^{k'}$ will be replaced by a symmetric polynomial,

$$\frac{1}{\binom{k}{k'}}\sum_{|S|=k'}\prod_{\ell\in S}z_{j\ell}$$

This gives us a map $\mathbb{C}_k[z_1,\ldots,z_n]\to\mathbb{C}_1[z_{11},\ldots,z_{nk}]$, and we also have a natural map in the other direction (replacing z_{ij} by z_i). So it suffices to check that both these transformations preserve stability, where one direction is straightforward (since it's just specialization). In this way, the general case can be reduced to the multi-affine case. We omit the details. □