Stanford University **Winter 2018**
Math 233A: Non-constructive methods in combinatorics **Instructor: Jan Vondrák**
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Lecture 18. Ramanujan Graphs continued

18.1 Interlacing polynomials

Last time, we proved the following lemma.

Lemma 18.1 *Two real-rooted, monic polynomials of the same degree, f and g, have a common interlacing if and only if for every $t \in [0,1]$, $h_t = tf + (1-t)g$ is real-rooted.*

We can extend this to multiple monic polynomials as follows. Suppose $f_1, f_2, ..., f_m$ are monic polynomials with the same degree n. Let R_k be the set of k-th largest roots of the polynomials (counting multiple roots multiple times). We say that the polynomials have a common interlacing if there exist $\gamma_n \leq \gamma_{n-1} \leq ... \leq \gamma_0$ such that for each k, $R_k \subseteq [\gamma_k, \gamma_{k-1}]$. Then it is not difficult to extend Lemma 18.1 to the following claim:

Claim 18.2 *Suppose $f_1, f_2, ..., f_m$ are monic polynomials of the same degree. Then the following are equivalent:*

1. *$f_1, f_2, ..., f_m$ have a common interlacing.*
2. *Each pair of polynomials f_i, f_j has a common interlacing.*
3. *There exists $i < j$ such that for the pair of polynomials f_i, f_j, and for any $t \in [0,1]$, $tf_i + (1-t)f_j$ is real rooted and has a common interlacing with the rest of the polynomials.*
4. *For any $\{t_i\}_{i=1}^m$ with each $t_i \geq 0$ and $\sum_i t_i = 1$, we have $\sum_i t_i f_i$ is real rooted.*

18.2 Interlacing families

Let S_i for i from 1 to m be finite sets with a corresponding distribution μ_i on them. We will be working with families of polynomials $f_{s_1,s_2,...,s_m}$, indexed by $s_i \in S_i$. We will define

$$f_{s_1,s_2,...,s_k}(x) = \mathop{\mathbb{E}}_{\substack{s_{k+1}\sim\mu_{k+1} \\ \vdots \\ s_m\sim\mu_m}} [f_{s_1,s_2,...,s_m}(x)]$$

We say that the family of polynomials forms an interlacing family if for any $k < m$, and $s_1 \in S_1$, $s_2 \in S_2$, ..., $s_k \in S_k$, the polynomials $\{f_{s_1,s_2,...,s_k,s}\}_{s\in S_{k+1}}$ have a common interlacing. This allows us to prove the following theorem.

Theorem 18.3 *[MSS '13] If $\{f_{s_1,s_2,...,s_m}\}$ form an interlacing family, then we can find $s_1 \in S_1$, $s_2 \in S_2$, ...,$s_m \in S_m$ such that*

$$\mathrm{maxroot}(f_{s_1,s_2,...,s_m}) \leq \mathrm{maxroot}(f_\emptyset(x)).$$

Proof: We will prove that for every $k \leq m$, and every $s_1, s_2, ..., s_{k-1}$, there exists $s_k \in S_k$ such that

$$\text{maxroot}(f_{s_1,s_2,...,s_k}) \leq f_{s_1,s_2,...,s_{k-1}}.$$

This is true because $f_{s_1,s_2,...,s_{k-1}}$ is a convex combination of $f_{s_1,s_2,...,s_{k-1},s_k}$ over various choices of s_k. This implies that the maximum root of $f_{s_1,s_2,...,s_{k-1}}$ is a convex combination of the maximum roots of $f_{s_1,s_2,...,s_{k-1},s_k}$ over various choices of s_k, in particular, its maximum root is larger than at least on one of the maximum roots of $f_{s_1,s_2,...,s_k}$. Applying this to each k, the theorem is proved. □

Let us now turn to adjacency matrices of graphs. Given a graph $G = (V, E)$ and a vector $s \in \{0,1\}^E$, the graph $G_s^{(2)}$ is the two-lift of G corresponding to the vector s, where a 0 coordinate indicates non-crossing edges and a 1 coordinate indicates crossing edges.

Then if A is the adjacency matrix of G, then A_s will be obtained from A by writing $(-1)^{s_e}$ for each spot corresponding to a 1 entry in A (recall that A has only 0 and 1 entries). Then let us denote the polynomial

$$\chi_{s(x)} = \det(xI - A_s).$$

We saw last time that

$$\chi_\emptyset(x) = \mathop{\mathbb{E}}_{s_1,s_2,...,s_m} [\chi_s(x)] = M_G(x),$$

where $M_G(x)$ denotes the matching polynomial of G. By Heilmann-Lieb, we know that the maximum root of the matching polynomial is at most $2\sqrt{d-1}$. We would like to conclude using Theorem 18.3 that this implies that there exists a choice of s such that the maximum root is $2\sqrt{d-1}$. We thus have to prove that the polynomials form an interlacing family, where we recall that we denote

$$\chi_{s_1,s_2...,s_k}(x) = \mathop{\mathbb{E}}_{s_{k+1},...,s_m} [\det(xI - A_s)].$$

First, let us prove the following:

Claim 18.4 *For $p_1, p_2, ..., p_m \in [0,1]$, define*

$$\chi_{p_1,p_2,...,p_m}(x) = \sum_{s_1,s_2,...,s_m \in \{0,1\}} \prod_{i:s_i=1} p_i \prod_{i:s_i=0} (1-p_i)\chi_s(x).$$

Then all such polynomials are real rooted.

This claim implies that the polynomials form an interlacing family. Indeed, it is easy to see that

$$\chi_{s_1,s_2,...,s_k}(x) = \chi_{s_1,s_2,...,s_k,\frac{1}{2},\frac{1}{2},...,\frac{1}{2}}(x).$$

Thus, this implies that any convex combination of $\chi_{s_1,s_2,...,s_{k-1},0}$ and $\chi_{s_1,s_2,...,s_{k-1},1}$ is real rooted (since it is equal to $\chi_{s_1,s_2,...,s_{k-1},t,\frac{1}{2},\frac{1}{2},...,\frac{1}{2}}$.

In order to prove the claim, we prove the following theorem:

Theorem 18.5 *If $\overrightarrow{v_1}$, $\overrightarrow{v_2}$, ..., $\overrightarrow{v_m}$ are random vectors in $\mathbb{C}^n$ with finite support, then*

$$\mathop{\mathbb{E}}_{v_1,v_2,...,v_m} \left[\det(xI - \sum_{i=1}^{m} v_i v_i^*)\right]$$

is a real stable polynomial.

For the case of Ramanujan graphs,

$$\chi_{s(x)} = \det(xI - A_s),$$

where we can write

$$A_s = dI - \sum_{e \in E} b_e b_e^*$$

for vectors

$$b_e = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ +1 \\ -1 \\ \vdots \\ 0 \end{pmatrix} \text{ if } s_e = 0, \qquad b_e = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ +1 \\ +1 \\ \vdots \\ 0 \end{pmatrix} \text{ if } s_e = 1.$$

Thus, Theorem 18.5 implies Claim 18.4. We will be using a few facts from linear algebra.

Fact 1 *If $A \in \mathbb{C}^{n \times n}$ is a nonsingular matrix, and $u, v \in \mathbb{C}^n$ are vectors, then*

$$(A + uv^*) = (1 + v^* A^{-1} u) \det(A). \tag{18.1}$$

Proof: First, assume $A = I$. Then we have

$$\begin{pmatrix} I & u \\ 0 & 1 + v^* u \end{pmatrix} = \begin{pmatrix} I & 0 \\ v^* & 1 \end{pmatrix} \begin{pmatrix} I + uv^* & u \\ 0 & 1 \end{pmatrix} \begin{pmatrix} I & 0 \\ -v^* & 1 \end{pmatrix}.$$

Taking the determinant of both sides, we obtain (18.1). In general, we have, for $u' = A^{-1}u$,

$$\det(A + uv^*) = \det(A) \det(I + u'v^*) = \det(A)(1 + v^* u') = (1 + v^* A^{-1} u) \det(A).$$

□

Recall that given a matrix A, the trace is the sum along the diagonal $\mathrm{Tr}(A) = \sum_i a_{ii}$. Also recall that for two matrices A and B,

$$\mathrm{Tr}(AB) = \sum_{i,j} a_{ij} b_{ji} = \mathrm{Tr}(BA).$$

Fact 2 *If $A, B \succeq 0$, then $\mathrm{Tr}(AB) \geq 0$.*

Proof: We can write $A = \sum_i \alpha_i u_i u_i^*$ for $\alpha_i \geq 0$, and $B = \beta_j \sum_j v_j v_j^*$ for $\beta_j \geq 0$. Then

$$\mathrm{Tr}(AB) = \sum_{i,j} \alpha_i \beta_j \,\mathrm{Tr}(u_i u_i^* v_j v_j^*) = \sum_{i,j} \alpha_i \beta_j \,\mathrm{Tr}(v_j^* u_i u_i^* v_j) = \sum_{i,j} \alpha_i \beta_j |v_j^* u_i|^2 \geq 0.$$

□

Fact 3 *For non-singular $A \in \mathbb{C}^{n \times n}$, and arbitrary $B \in \mathbb{C}^{n \times n}$, we have*

$$\frac{d}{dt} \left(\det(A + tB)\right)\Big|_{t=0} = \det(A) \,\mathrm{Tr}(A^{-1}B).$$

Proof: We may assume that $A = I$. Let $B = \sum_{i,j} b_{ij} e_i e_j^*$. Then

$$\frac{d}{dt}\left(\det(I + t\sum_{i,j} b_{ij} e_i e_j^*)\right)\Bigg|_{t=0} = \sum_{i,j} \frac{\partial}{\partial t_{ij}}\left(\det(I + \sum_{i',j'} t_{i'j'} b_{i'j'} e_{i'} e_{j'}^*)\right)\Bigg|_{t_{ij}=0}$$

$$= \sum_{i,j} \frac{\partial}{\partial t_{ij}}\left(\det(I + t_{ij} b_{ij} e_i e_j^*)\right)\Bigg|_{t_{ij}=0} = \sum_{i,j} \frac{\partial}{\partial t_{ij}}(1 + t_{ij} b_{ij} \delta_{i,j})\Bigg|_{t_{ij}=0} = \sum_i b_{ii} = \mathrm{Tr}(B).$$

□

Proof:[of Theorem 18.5, simplified by James Lee] We first prove the following lemma

Lemma 18.6 *If $A \in \mathbb{C}^{n\times n}$ is a fixed matrix, and $v \in \mathbb{C}^n$ is a random vector, then*

$$\mathbb{E}\left[\det(A - vv^*)\right] = \left(1 - \frac{d}{dt}\right)\det\left(A + t\,\mathbb{E}_v\left[vv^*\right]\right)\Bigg|_{t=0}.$$

Proof: First, assume A is non-singular. Then by (18.1) we have

$$\det(A - vv^*) = \det(A)(1 - v^* A^{-1} v) = \det(A)(1 - \mathrm{Tr}(A^{-1} vv^*).$$

Thus,

$$\mathbb{E}_v\left[\det(A - vv^*)\right] = \det(A)(1 - \mathrm{Tr}(A^{-1}\,\mathbb{E}[vv^*]))$$

By Fact 3, we have

$$\frac{d}{dt}\left(\det(A + t\,\mathbb{E}[vv^*])\right)\Bigg|_{t=0} = \det(A)\,\mathrm{Tr}(A^{-1}\,\mathbb{E}[vv^*]),$$

which proves the lemma for non-singular A. For singular A, we can take a sequence $A_\epsilon \to A$ of nonsingular matrices. The equation will hold for each A_ϵ, and since both sides of the equation are continuous, the lemma must also hold for A. □

For the proof of the theorem, we need one more claim

Claim 18.7 *The operator $\left(1 - \frac{\partial}{\partial z_i}\right)$ preserves stability of $f(z_1, z_2, ..., z_m)$.*

Proof: Next time. □

We have, for independent vectors $v_1, v_2, ..., v_m$,

$$\mathbb{E}_{v_1,\dots,v_m}\left[\det(A - \sum_i v_i v_i^*)\right] = \left(1 - \frac{\partial}{\partial t_1}\right)\dots\left(1 - \frac{\partial}{\partial t_m}\right)\det(A + \sum_i t_i\,\mathbb{E}[v_i v_i^*])\Bigg|_{t_1=0,\dots,t_m=0}.$$

We thus have

$$\mathbb{E}\left[xI - \sum_i v_i v_i^*\right] = \left(1 - \frac{\partial}{\partial t_1}\right)\dots\left(1 - \frac{\partial}{\partial t_m}\right)\det(xI + \sum_i t_i\,\mathbb{E}[v_i v_i^*])\Bigg|_{t_1=0,\dots,t_m=0}.$$

We know that

$$\det(xI + \sum_i t_i \, \mathbb{E}[v_i v_i^*])$$

is stable because each $\mathbb{E}[v_i v_i^*]$ is positive semidefinite (using a theorem from Lecture 13). Since the operators preserve stability, we thus have that

$$\mathbb{E}\left[\det(xI - \sum_i v_i v_i^*)\right]$$

is stable, and so the theorem is proved. $\square$