MATH 233A: Non-constructive Methods in Combinatorics, Spring 2018
HOMEWORK 1
Due Tuesday, February 6

Please try to solve the homework on your own. Discussions are okay but make your own effort.

Problem 1. (a) Prove by induction that the number of trees on vertices $\{1, 2, \ldots, t\}$ with degrees $d_1, \ldots, d_t \geq 1$ such that $\sum_{i=1}^{t} d_i = 2(t-1)$ is the multinomial coefficient

$$\binom{t-2}{d_1 - 1, d_2 - 1, \ldots, d_t - 1}.$$

(b) Let F be a fixed forest in K_n with connected components of vertex-sizes $f_1, f_2, \ldots, f_t$. Prove that the number of spanning trees containing F is

$$n^{t-2} \prod_{i=1}^{t} f_i.$$

(c) Show that this implies

$$\Pr[A \subseteq T \ \& \ B \subseteq T] = \Pr[A \subseteq T] \cdot \Pr[B \subseteq T]$$

where A, B are fixed vertex-disjoint sets of edges in K_n and T is a uniformly random spanning tree in K_n.

Problem 2. Let G be a d-regular bipartite graph of girth (minimum cycle length) at least g, such that $d \leq 2^{g/2}$, and let's say $g \geq 16$ (sufficiently large constant). Prove that there is an "acyclic edge coloring" with $2d$ colors: incident edges get different colors and every cycle gets at least 3 different colors.

Hint: Start with the fact that every d-regular bipartite graph has an edge coloring with d colors.

Problem 3. Prove that for any fixed $p \in (0,1)^n$ satisfying Shearer's conditions ($q_I(p) > 0$ for every $I \in Ind(G)$),

(a) the $\breve{q}_S(p)$ polynomials are log-submodular in S: $\breve{q}_{S \cup T}(p) \breve{q}_{S \cap T}(p) \leq \breve{q}_S(p) \breve{q}_T(p)$.

Hint: Prove that $\breve{q}_S / \breve{q}_{S-a} \geq \breve{q}_T / \breve{q}_{T-a}$, whenever $a \in S \subset T$.

(b) the $q_S(p)$ polynomials are also log-submodular: $q_{S \cup T}(p) q_{S \cap T}(p) \leq q_S(p) q_T(p)$.

Hint: Use part (a).

Problem 4. Let $G = (V, E)$ be a cycle on tn vertices, and $V = V_1 \cup \ldots \cup V_n$ a partition of the vertices into (not necessarily consecutive) sets of size $|V_i| = t$. Prove for $t = 11$ and every $n \geq 1$ that there is an independent set containing one vertex from each set V_i.

Bonus question: What is the smallest t for which you can prove this?