MATH 233A: Non-constructive Methods in Combinatorics, Spring 2018
HOMEWORK 2
Due Tuesday, February 20

Please try to solve the homework on your own. Discussions are okay but make your own effort.

Problem 1. Prove from the Cluster Expansion Lemma (not using Shearer's lemma!) that

$$p(1 + ed) \le 1$$

is a sufficient condition to avoid all bad events in the symmetric case.

Problem 2. True or false? In the Moser-Tardos algorithm, for a fixed $t > 1$ and $i \in [n]$, the probability that the t-th resampled event is $\mathcal{E}_i$ is at most $\Pr[\mathcal{E}_i]$. Prove this or find a counterexample.

Problem 3. Let the dependency graph be a tree T, with (an arbitrarily chosen) root r. Prove that p satisfies Shearer's conditions, if and only if there are parameters $z_v \in (0, 1)$ such that $p_v = z_v \prod_{w \in C(v)} (1 - z_w)$, where $C(v)$ are the children of v (neighbors not on the path to r).
Hint: Find an expression for z_v in terms of Shearer's polynomials.

Problem 4. Consider a 4-partite graph on $V_0 \cup V_1 \cup V_2 \cup V_3$ such that $e(V_i, V_{i+1 \bmod 4}) \ge \rho |V_i||V_{i+1 \bmod 4}|$ for each i. Prove that if $\rho \ge 1/\sqrt{2}$, then there must exist C_4 as a subgraph with one vertex in each V_i.

Bonus question: Can you identify the optimal threshold ρ for this question?