MATH 233A: Non-constructive Methods in Combinatorics, Winter 2018
HOMEWORK 3
Due Tuesday, March 13

Please try to solve the homework on your own. Discussions are okay but make your own effort.

Problem 1. Prove that if A, B are Hermitian matrices such that $A - B$ is positive semidefinite, then the k-th top eigenvalue of A is at least the k-th top eigenvalue of B.

Problem 2. Use the matrix-tree formula to deduce that the number of spanning trees in K_n is n^{n-2} (Cayley's formula).

Problem 3. Show that for every tree T, the matching defect polynomial

$$\mathcal{M}_T(x) = \sum_{\text{matching } M \subset T} (-1)^{|M|} x^{n-2|M|}$$

is equal to the characteristic polynomial of its adjacency matrix,

$$\chi_A(x) = \det(xI - A),$$

where $A_{ij} = 1$ if $(i, j) \in E(T)$ and 0 otherwise.

Problem 4. Prove that if $\lambda_1, \ldots, \lambda_n \geq 0$ are positive constant parameters and $1 \leq k \leq n$, then

$$f(z_1, \ldots, z_n) = \sum_{S \in \binom{[n]}{k}} \prod_{j \in S} (1 - \lambda_j z_j)$$

is a stable polynomial.