MATH 233B: Polyhedral techniques in combinatorial optimization
HOMEWORK 1
Due February 7, 2017

You can discuss the problems but try to solve them on your own. If you collaborate on a problem, please acknowledge it on your homework. Don't use any books or the internet.

Problem 1. Prove that the edges of a d-regular bipartite graph (every vertex has degree d) can be colored with d colors so that every vertex is incident to all d colors. (Use König's theorem.)

Problem 2. Let G is a bipartite graph with parts A, B. Let $S \subseteq A$ be such that there is a matching covering S, and $T \subseteq B$ such that there is a matching covering T. Prove that there is a matching covering both S and T.

Problem 3. Let $P_{match}(G) = \text{conv}\{\chi_M : M \text{ is a matching in } G\}$ denote the matching polytope of G. Prove that $P_{match}(G) \cap \{\mathbf{x} : \mathbf{1}^T\mathbf{x} = k\}$ is the convex hull of all matchings of size exactly k.

Problem 4. Consider the perfect matching polytope, $P_{perf}(G)$. An edge is a line segment between two vertices $s = [\chi_M, \chi_N]$ such that $s = H \cap P_{perf}(G)$ for some hyperplane $H = \{\mathbf{x} : \mathbf{w}^T\mathbf{x} = \lambda\}$ and $\mathbf{w}^T\mathbf{x} \leq \lambda$ for all $\mathbf{x} \in P$. Prove that $[\chi_M, \chi_N]$ is an edge if and only if $M \Delta N$ is a single cycle. What is the certifying hyperplane?

Bonus problem. Let $D = (V, A)$ be a directed graph and $T = (V, B)$ a directed tree (some orientation of a spanning tree on V). Define a matrix $C \in \mathbb{R}^{A \times B}$, defined as follows. For an edge $a = (u, v)$, let P_a be the unique path from u to v in T. For $b \in B$, we define

- $C_{a,b} = 1$ if b occurs in forward direction on P_a,
- $C_{a,b} = -1$ if b occurs in backward direction on P_a,
- $C_{a,b} = 0$ if $b \notin P_a$.

Prove that C is a totally unimodular matrix.