MATH 233B: Polyhedral techniques in combinatorial optimization
HOMEWORK 2
Due February 28

You can discuss the problems but try to solve them on your own. If you collaborate on a problem, please acknowledge it on your homework. Don't use any books or the internet.

Problem 1. Let $G = (V, E)$ be a 3-regular graph and

$$P = \text{conv}(\{\chi_F : F \subseteq E \text{ is a collection of vertex-disjoint cycles covering all vertices in } V\}).$$

What is the description of P in terms of linear inequalities? Prove that the vertices of your description are exactly the vectors χ_F described above.

Problem 2. A *circuit* in a matroid $\mathcal{M}$ is an inclusion-wise minimal dependent set, and a *cut* is an inclusion-wise minimal set which intersects every base. Prove that if a circuit C intersects a cut D, then $|C \cap D| \geq 2$. (*Hint:* use the strong exchange property, or reasoning similar to its proof.)

Problem 3. Prove that $[\chi_B, \chi_C]$ is an edge of the matroid base polytope if and only if B, C are bases such that $|B \Delta C| = 2$.

Problem 4. Let A be a non-singular matrix. Prove that for any choice of a subset of rows I, there is a subset of columns J, such that the matrices corresponding to $I \times J$ and $\overline{I} \times \overline{J}$ are both non-singular. (*Hint:* matroid intersection and duality are useful here.)

Problem 5. Consider the following graph orientation problem: we would like to orient the edges of a graph G in such a way that each vertex has at most k incoming edges. Prove that this is possible if and only if $|E[W]| \leq k|W|$ for each subset of vertices W.

Bonus problem. Given a graph G, two players play the following game. *Destroyer* plays first and *removes* an edge of his choice. Then *Connector* takes his turn and *fixes* an edge of his choice. A removed edge cannot be fixed anymore and a fixed edge cannot be removed anymore. The players alternate, one edge at a time. *Connector* wins if he fixes a spanning tree in the graph. *Destroyer* wins if he manages to disconnect the graph. (It's easy to see that exactly one player wins at some point.) Prove that *Connector* has a winning strategy, if and only if G contains two edge-disjoint spanning trees.