

1 Matroid intersection

Given two matroids $\mathcal{M}_1 = (E, \mathcal{I}_1)$ and $\mathcal{M}_2 = (E, \mathcal{I}_2)$ over the same ground set, we are interested in their intersection, defined as

$$\mathcal{M}_1 \cap \mathcal{M}_2 = (E, \mathcal{I}_1 \cap \mathcal{I}_2).$$

In general, $\mathcal{M}_1 \cap \mathcal{M}_2$ is not a matroid. Specifically, the greedy algorithm cannot be used to solve the maximal weight common independent set problem. In this lecture, we show that nevertheless, there is a polynomial time algorithm to solve this problem.

Example 1 (Bipartite matching). The matching problem for a bipartite graph $(V_1 \cup V_2, E \subseteq V_1 \times V_2)$ can be written as matroid intersection using

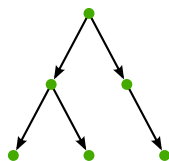
$$\begin{aligned} \mathcal{I}_1 &= \{F \subseteq E \mid \forall v \in V_1 : |F \cap \delta(v)| \leq 1\} \\ \mathcal{I}_2 &= \{F \subseteq E \mid \forall v \in V_2 : |F \cap \delta(v)| \leq 1\}. \end{aligned}$$

Here, \mathcal{I}_i are all edge collections such that each node in V_i has at most one incident edge. The intersection $\mathcal{I}_1 \cap \mathcal{I}_2$ is the set of all matchings.

Example 2 (Arborescences). Consider a directed graph $D = (V, E \subseteq V \times V)$. A set $T \subseteq E$ is an *arborescence* (oriented forest) if

1. T does not contain a cycle (ignoring directions of edges).
2. Every vertex in V has at most one incoming edge.

An arborescence T with $|T| = n - 1$ will have one incoming edge incident on each node except one. If we denote this special node as root, this is an oriented spanning tree as shown in the figure.



Letting $\mathcal{M}_1 = (E, \mathcal{I}_1)$ be the graphic matroid that ensures condition 1, and $\mathcal{M}_2 = (E, \mathcal{I}_2)$ the partition matroid that ensures condition 2, the set of arborescences is given by $\mathcal{I}_1 \cap \mathcal{I}_2$.

2 The matroid intersection polytope

Definition 3. For matroids $\mathcal{M}_1 = (E, \mathcal{I}_1)$ and $\mathcal{M}_2 = (E, \mathcal{I}_2)$, define the *matroid intersection polytope* $\mathcal{P}(\mathcal{M}_1 \cap \mathcal{M}_2) = \text{conv}\{\chi_I \mid I \in \mathcal{I}_1 \cap \mathcal{I}_2\}$.

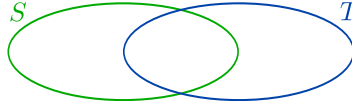
Theorem 4 (Edmonds).

$$\mathcal{P}(\mathcal{M}_1 \cap \mathcal{M}_2) = \mathcal{P}(\mathcal{M}_1) \cap \mathcal{P}(\mathcal{M}_2).$$

Before we prove the theorem we need three lemmas.

Definition 5. Let \mathcal{M} be a matroid with rank function r . Let $\mathbf{x} \in \mathcal{P}(\mathcal{M})$. We call a set S a *tight set* of \mathbf{x} with respect to r if $x(S) = r(S)$.

Lemma 6 (Uncrossing operation). Let \mathcal{M} be a matroid with rank function r , and let $\mathbf{x} \in \mathcal{P}(\mathcal{M})$. If S and T are tight sets of \mathbf{x} with respect to r , then so are $S \cup T$ and $S \cap T$.



Proof. Since $\mathbf{x} \in \mathcal{P}(\mathcal{M})$, we have $r(S \cup T) \geq x(S \cup T)$ and $r(S \cap T) \geq x(S \cap T)$. Therefore,

$$\begin{aligned} r(S \cup T) + r(S \cap T) &\geq x(S \cup T) + x(S \cap T) \\ &\stackrel{(a)}{=} x(S) + x(T) \\ &\stackrel{(b)}{=} r(S) + r(T) \\ &\stackrel{(c)}{\geq} r(S \cup T) + r(S \cap T), \end{aligned}$$

where (a) is due to linearity of $x(A) = \sum_{i \in A} x_i$, (b) is due to the tightness of S and T , and (c) is because r is submodular. Therefore all inequalities must be equalities. \square

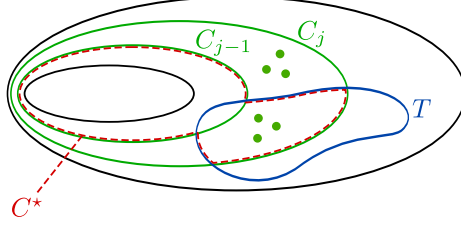
Lemma 7. Let \mathcal{M} be a matroid with rank function r , and let $\mathbf{x} \in \mathcal{P}(\mathcal{M})$. Let $\mathcal{C} = \{C_1, \dots, C_k\}$ with $\emptyset \subset C_1 \subset \dots \subset C_k$ be an inclusionwise maximal chain of tight sets of \mathbf{x} with respect to r . Then every tight set T of \mathbf{x} with respect to r must satisfy $\chi_T \in \text{span}\{\chi_C : C \in \mathcal{C}\}$.

Proof. Suppose there is some tight set T with $\chi_T \notin \text{span}\{\chi_C : C \in \mathcal{C}\}$.

1. If $T \not\subseteq C_k$, then let $C^* = C_k \cup T$. We have $C_k \subset C^*$ and C^* is tight by Lemma 6. Therefore it can be appended to \mathcal{C} , a contradiction to \mathcal{C} being maximal.
2. Now consider $T \subseteq C_k$. We argue that T must cut one of the “increment sets” $C_j \setminus C_{j-1}$ into two non-empty parts. (By convention, $C_0 = \emptyset$.) Suppose first that this was not true. Then we can write $T = \bigcup_{j \in \mathcal{J}} (C_j \setminus C_{j-1})$ for some $\mathcal{J} \subseteq \{1, \dots, k\}$. This implies

$$\chi_T = \sum_{j \in \mathcal{J}} \chi_{C_j \setminus C_{j-1}} = \sum_{j \in \mathcal{J}} (\chi_{C_j} - \chi_{C_{j-1}}),$$

which contradicts the assumption that $\chi_T \notin \text{span}\{\chi_C : C \in \mathcal{C}\}$. Therefore $T \cap (C_j \setminus C_{j-1})$ is a proper subset of $C_j \setminus C_{j-1}$ for some j and we must have the situation depicted here.



Let $C^* = (T \cap C_j) \cup C_{j-1}$, which is tight by Lemma 6. By construction, $C_{j-1} \subset C^* \subset C_j$, and C^* can be inserted in the chain \mathcal{C} , a contradiction to maximality. □

Lemma 8. Let \mathcal{M}_1 and \mathcal{M}_2 be two matroids over the same ground set with rank functions r_1 and r_2 . If \mathbf{x} is a vertex of $\mathcal{P}(\mathcal{M}_1) \cap \mathcal{P}(\mathcal{M}_2)$, then there exist two chains

$$\begin{aligned} \mathcal{C} &= \{C_1, \dots, C_k\} & \text{with } \emptyset \subset C_1 \subset C_2 \subset \dots \subset C_k, \\ \mathcal{D} &= \{D_1, \dots, D_l\} & \text{with } \emptyset \subset D_1 \subset D_2 \subset \dots \subset D_l, \end{aligned}$$

such that

1. $\forall i; x(C_i) = r_1(C_i)$ and $\forall j; x(D_j) = r_2(D_j)$,
2. $\{\chi_C \mid C \in \mathcal{C} \cup \mathcal{D}\}$ contains $|\text{supp}(\mathbf{x})|$ linearly independent vectors.

Proof. Construct \mathcal{C} and \mathcal{D} as inclusionwise maximal chains of tight sets of \mathbf{x} with respect to r_1 and r_2 , correspondingly. Then the first part of the lemma is true by construction. Recall that

$$\begin{aligned} \mathcal{P}(\mathcal{M}_1) \cap \mathcal{P}(\mathcal{M}_2) &= \{\mathbf{x} \geq 0 : \\ &\quad \forall S; x(S) \leq r_1(S), \\ &\quad \forall S; x(S) \leq r_2(S)\}. \end{aligned}$$

Since \mathbf{x} is a vertex of this polytope, it must have $|E|$ linearly independent tight constraints. Exactly $|E| - |\text{supp}(\mathbf{x})|$ of them are the constraints $x_i = 0$. To account for the remaining constraints, there must exist $|\text{supp}(\mathbf{x})|$ linearly independent vectors χ_T such that T is a tight set of \mathbf{x} with respect to r_1 or r_2 .

Lemma 7 implies that $\mathcal{S}_1 = \text{span}\{\chi_C : C \in \mathcal{C}\}$ contains the characteristic vectors χ_T of all tight sets T of \mathbf{x} with respect to r_1 . Likewise, for $\mathcal{S}_2 = \text{span}\{\chi_D : D \in \mathcal{D}\}$, with respect to r_2 . It follows that $\text{span}\{\chi_C : C \in \mathcal{C} \cup \mathcal{D}\} \supseteq \mathcal{S}_1 \cup \mathcal{S}_2$ contains the characteristic vectors χ_T of all tight sets T of \mathbf{x} with respect to r_1 or r_2 . Therefore, the two chains span a subspace of dimension at least $|\text{supp}(\mathbf{x})|$. □

Proof of Theorem 4. The inclusion $\mathcal{P}(\mathcal{M}_1 \cap \mathcal{M}_2) \subseteq \mathcal{P}(\mathcal{M}_1) \cap \mathcal{P}(\mathcal{M}_2)$ follows from the definition. We show the reverse inclusion by induction on $|E|$. Clearly, it holds for $|E| = 0$.

Assume \mathbf{x} is a vertex of $\mathcal{P}(\mathcal{M}_1) \cap \mathcal{P}(\mathcal{M}_2)$ that is not in $\mathcal{P}(\mathcal{M}_1 \cap \mathcal{M}_2)$.

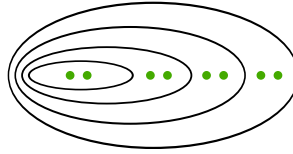
1. If $x_i = 0$ for some i , then delete i from \mathcal{M}_1 and \mathcal{M}_2 and remove the i th coordinate of \mathbf{x} to obtain $\tilde{\mathbf{x}}$. We have $\tilde{\mathbf{x}} \in \mathcal{P}(\mathcal{M}_1 \setminus i)$ and $\tilde{\mathbf{x}} \in \mathcal{P}(\mathcal{M}_2 \setminus i)$. From induction, we have $\tilde{\mathbf{x}} \in \mathcal{P}((\mathcal{M}_1 \setminus i) \cap (\mathcal{M}_2 \setminus i))$, which implies $\mathbf{x} \in \mathcal{P}(\mathcal{M}_1 \cap \mathcal{M}_2)$.

2. If $x_i = 1$ for some i , then contract i from \mathcal{M}_1 and \mathcal{M}_2 and remove the i -th coordinate of \mathbf{x} to obtain $\tilde{\mathbf{x}}$. We have

$$\begin{aligned} \forall S \subseteq E - i; \quad \tilde{x}(S) &= x(S + i) - 1 \\ &\leq r_1(S + i) - 1 \\ &= r_{\mathcal{M}_1/i}(S), \end{aligned}$$

where the last equality is from the definition of contraction. Hence, $\tilde{\mathbf{x}} \in \mathcal{P}(\mathcal{M}_1/i)$, and likewise $\tilde{\mathbf{x}} \in \mathcal{P}(\mathcal{M}_2/i)$. By induction, this implies $\tilde{\mathbf{x}} \in \mathcal{P}((\mathcal{M}_1/i) \cap (\mathcal{M}_2/i))$, i.e., $\tilde{\mathbf{x}}$ is a convex combination of $\{\chi_I : I \text{ is independent in } \mathcal{M}_1/i, \mathcal{M}_2/i\}$. Adding i to the independent sets in $\mathcal{M}_1/i, \mathcal{M}_2/i$ results in independent sets in $\mathcal{M}_1, \mathcal{M}_2$, and thus \mathbf{x} is a convex combination of $\{\chi_I : I \text{ is independent in both } \mathcal{M}_1, \mathcal{M}_2\}$, or equivalently, $\mathbf{x} \in \mathcal{P}(\mathcal{M}_1 \cap \mathcal{M}_2)$.

3. If $0 < x_i < 1$ for all i , then apply Lemma 8 to \mathbf{x} to construct chains \mathcal{C} and \mathcal{D} . Note that $x(C_i) = r(C_i)$ is an integer for all i . Since every x_i is fractional, the increment sets $C_i \setminus C_{i-1}$ must each contain at least two elements to ensure integer sums $x(C_i)$, as shown here.



Hence $|\mathcal{C}| \leq |E|/2$, and likewise $|\mathcal{D}| \leq |E|/2$. The lemma states that $\{\chi_C \mid C \in \mathcal{C} \cup \mathcal{D}\}$ contains $\text{supp}(\mathbf{x}) = |E|$ linearly independent vectors. Therefore, we must have $|\mathcal{C}| = |\mathcal{D}| = |E|/2$. This however implies that the maximal sets in the two chains must be $C_k = D_l = E$, i.e., $\mathcal{C} \cup \mathcal{D}$ contains at most $|E| - 1$ sets, a contradiction.

□

Remark 9. Theorem 4 implies

$$\begin{aligned} \mathcal{P}(\mathcal{M}_1 \cap \mathcal{M}_2) &= \{x \mid x \geq 0, \\ &\quad \forall S : x(S) \leq r_1(S), \\ &\quad \forall S : x(S) \leq r_2(S)\}. \end{aligned}$$

In other words, the matroid intersection polytope has an efficient separation oracle which consists of sequentially checking both \mathcal{M}_1 and \mathcal{M}_2 separation oracles. Using the ellipsoid method to convert a separation oracle into an optimization algorithm allows us to construct a polynomial-time algorithm for optimization over $\mathcal{P}(\mathcal{M}_1 \cap \mathcal{M}_2)$. There are combinatorial algorithms for matroid intersection that are more practical than this construction, but we will not cover them here.

Remark 10. Theorem 4 implies that the set of vertices of $\mathcal{P}(\mathcal{M}_1 \cap \mathcal{M}_2)$ is contained in $\{\chi_I \mid I \in \mathcal{I}_1 \cap \mathcal{I}_2\}$. The two sets are in fact equal. This is easy to see, since no point in $\{0, 1\}^n$ can be a convex combination of other such points.

3 A min-max formula for matroid intersection

LP duality implies a min-max relation for the problem $\max\{\mathbf{w}^T \mathbf{x} : \mathbf{x} \in P(\mathcal{M}_1) \cap P(\mathcal{M}_2)\}$. In case $\mathbf{w} = \mathbf{1}$, the min-max formula becomes in fact much simpler.

Theorem 11 (The matroid intersection theorem). For any two matroids $\mathcal{M}_1 = (E, \mathcal{I}_1)$, $\mathcal{M}_2 = (E, \mathcal{I}_2)$,

$$\max_{I \in \mathcal{I}_1 \cap \mathcal{I}_2} |I| = \min_{A \subseteq E} (r_1(A) + r_2(E \setminus A)).$$

Proof. The inequality $\max_{I \in \mathcal{I}_1 \cap \mathcal{I}_2} |I| \leq \min_{A \subseteq E} (r_1(A) + r_2(E \setminus A))$ is easy to see, because for any common independent set I and any $A \subseteq E$, we have

$$|I| = |I \cap A| + |I \cap (E \setminus A)| \leq r_1(A) + r_2(E \setminus A).$$

To prove the reverse inequality, we need to exhibit a common independent set of size $k = \min_{A \subseteq E} (r_1(A) + r_2(E \setminus A))$. Let us proceed by induction on the number of elements such that $r_1(\{i\}) = r_2(\{i\}) = 1$.

1. If there is no such element, we can partition the ground set so that $A = \{i : r_1(\{i\}) = 0\}$ and for all $j \in E \setminus A$, $r_2(\{j\}) = 0$. Hence, $r_1(A) + r_2(E \setminus A) = 0 + 0 = 0$ and there is nothing to prove.
2. Otherwise, assume that $r_1(\{e\}) = r_2(\{e\}) = 1$. First, delete e from both matroids and apply the induction hypothesis to $(\mathcal{M}_1 \setminus e) \cap (\mathcal{M}_2 \setminus e)$. (Note that the rank functions after deletion are unchanged, except for restriction to $E - e$.) If $\min_{A \subseteq E - e} (r_1(A) + r_2((E - e) \setminus A)) \geq k$, then by induction $\mathcal{M}_1 \setminus e$ and $\mathcal{M}_2 \setminus e$ contain a common independent set of size k and we are done because this is also a common independent set in $\mathcal{M}_1 \cap \mathcal{M}_2$. So we can assume that there is a set $A \subseteq E - e$ such that

$$r_1(A) + r_2((E - e) \setminus A) \leq k - 1. \quad (1)$$

Next, we contract e in $\mathcal{M}_1, \mathcal{M}_2$ to obtain $\mathcal{M}_1/e, \mathcal{M}_2/e$. By induction, if $\min_{B \subseteq E - e} (r_{\mathcal{M}_1/e}(B) + r_{\mathcal{M}_2/e}((E - e) \setminus B)) \geq k - 1$, then $(\mathcal{M}_1/e) \cap (\mathcal{M}_2/e)$ contains a common independent set of size $k - 1$. By the properties of contraction, we would be done because the element e can be added to this set, to obtain a common independent set in $\mathcal{M}_1 \cap \mathcal{M}_2$ of size k . Hence, we can assume that there is a set $B \subseteq E - e$ such that $r_{\mathcal{M}_1/e}(B) + r_{\mathcal{M}_2/e}((E - e) \setminus B) \leq k - 2$. The rank functions after contraction are $r_{\mathcal{M}_1/e}(B) = r_1(B + e) - 1$, and $r_{\mathcal{M}_2/e}((E - e) \setminus B) = r_2(E \setminus B) - 1$. So we assume that

$$r_1(B + e) + r_2(E \setminus B) \leq k. \quad (2)$$

Let us add up these two inequalities and use submodularity of the rank functions r_1, r_2 :

$$\begin{aligned} 2k - 1 &\geq (r_1(A) + r_1(B + e)) + (r_2((E - e) \setminus A) + r_2(E \setminus B)) \\ &\geq (r_1(A \cup B + e) + r_1(A \cap B)) + (r_2(E \setminus (A \cap B)) + r_2(E \setminus (A \cup B + e))) \end{aligned}$$

where we used the fact that $e \notin A \cup B$. On the other hand, by our definition of k , we have $r_1(A \cap B) + r_2(E \setminus (A \cap B)) \geq k$, and also $r_1(A \cup B + e) + r_2(E \setminus (A \cup B + e)) \geq k$. This is a contradiction.

□