

## 1 Matroid intersection algorithmically

Last time we saw the following min-max formula: For any two matroids  $\mathcal{M}_1 = (E, \mathcal{I}_1), \mathcal{M}_2 = (E, \mathcal{I}_2)$ ,

$$\max_{I \in \mathcal{I}_1 \cap \mathcal{I}_2} |I| = \min_{S \subseteq E} (r_1(S) + r_2(E \setminus S)).$$

Today we will describe an algorithm that can actually solve this maximization problem. The first idea is to start with  $I = \emptyset$ , and then try to augment it. For this, we can define

$$X_1^I = \{e \in E \setminus I : I + e \in \mathcal{I}_1\} \quad X_2^I = \{e \in E \setminus I : I + e \in \mathcal{I}_2\}$$

If  $X_1 \cap X_2 \neq \emptyset$ , then we can add any  $e \in X_1 \cap X_2$ . So we might as well assume that  $X_1 \cap X_2 = \emptyset$ . Just as in the case of bipartite matching, we'll want to use some analogy of augmenting paths. So we can define the allowable exchanges

$$\mathcal{E}_1^I = \{(i, j) : i \in I, j \in E \setminus I, I + j - i \in \mathcal{I}_1\} \quad \mathcal{E}_2^I = \{(j, i) : i \in I, j \in E \setminus I, I + j - i \in \mathcal{I}_2\}$$

Our hope is to follow a sequence of legal exchanges and thus increase the size of  $I$ . However, note that unlike in the bipartite matching case, it's not clear that just because each single exchange is allowable, then doing them all together maintains independence. Indeed, by considering e.g. the graphic matroid on  $K_4$ , we see that this can indeed be a problem.

So the actual algorithm is as follows.

*Initialize  $I := \emptyset$ . As long as there exists a directed path  $X_1^I \rightarrow X_2^I$  in  $\mathcal{E}_1^I \cup \mathcal{E}_2^I$ , choose such a path  $P$  of minimum length and update  $I$  by*

$$I := I \Delta V(P).$$

We need to show two claims.

**Claim 1.** If there is no  $X_1^I \rightarrow X_2^I$  path, then  $I$  is optimal (i.e.  $|I|$  is maximal).

**Claim 2.** If  $P$  is a shortest-length  $X_1^I \rightarrow X_2^I$  path then  $I \Delta V(P) \in \mathcal{I}_1 \cap \mathcal{I}_2$ .

**Proof of Claim 1.** We work in the directed graph whose edges are  $\mathcal{E}_1 \cup \mathcal{E}_2$ . Suppose there is no directed path  $X_1 \rightarrow X_2$ . Set

$$U = \{e : \exists \text{ directed path } e \rightarrow X_2\}.$$

We claim that  $r_1(U) = |I \cap U|$ . For if not, then we could extend  $I \cap U$  inside  $U$  by some  $u \in U \setminus I$  such that  $(I \cap U) + u \in \mathcal{I}_1$ . If  $I \subseteq U$  this would mean that  $u \in X_1$  which implies that there is a

directed path from  $X_1$  to  $X_2$ , a contradiction. Else, if  $I \setminus U \neq \emptyset$ , by the extension axiom, we can extend  $(I \cap U) + u$  by elements of  $I \setminus U$  until we reach the cardinality of  $I$ . So there is exactly one element  $v \in I \setminus U$  that we are missing, meaning that we've extended to an independent set  $(I - v) + u \in \mathcal{I}_1$ . But that means that we have an edge  $(v, u) \in \mathcal{E}_1$ , which, when combined with a directed path from  $u$  to  $X_2$ , implies that  $v \in U$ , a contradiction.

Similarly, we can prove that  $r_2(E \setminus U) = |I \setminus U|$ . Therefore, we get that

$$|I| = r_1(U) + r_2(E \setminus U)$$

and this means that we get a certificate of maximality, by the (weak version of the) min-max formula.  $\square$

**Lemma 1 (Strong exchange property)** *If  $B_1, B_2$  are bases in a matroid  $\mathcal{M}$  with  $B_1 \neq B_2$ , then for any  $y \in B_2 \setminus B_1$  there is  $x \in B_1 \setminus B_2$  so that  $B_1 - x + y, B_2 - y + x$  are both bases.*

**Proof:** For any  $y \in B_2 \setminus B_1$ , we get that  $B_1 + y \notin \mathcal{I}$ . So let  $C$  be an inclusion-wise minimal dependent subset of  $B_1 + y$  (a circuit). If  $x \in C - y$ , then  $C - x$  is independent by minimality, so we can extend it from  $B_1$  to a base, and in fact this base must be  $B_1 - x + y$ , because we cannot add  $x$  and we need to add all the other elements to achieve the cardinality of  $B_1$ . So any element of  $C - x$  is a possible exchange for  $y$  in  $B_1$ .

For  $B_2$ , we first know that  $y \in \text{span}(C - y)$ . Therefore,

$$\text{rank}((B_2 - y) \cup (C - y)) = \text{rank}(B_2 \cup C)$$

which is full rank in  $\mathcal{M}$ , since  $B_2$  is a base. Therefore,  $B_2 - y$  can be extended to a base within  $(B_2 - y) \cup (C - y)$ . We are adding only 1 element here and this element must be in  $C - y$ , hence we find  $x \in C - y$  such that  $B_2 - y + x$  is a base.  $\square$

**Lemma 2** *For any  $I, J \in \mathcal{I}_1$ ,  $|I| = |J|$ , there is a perfect matching between  $I \setminus J$  and  $J \setminus I$  in  $\mathcal{E}_1^I$ .*

**Proof:** First, assume that  $I, J$  are bases (by truncating the matroid to sets of size  $\leq |I|$ , i.e. declaring all larger sets to be dependent). By strong exchange, we can pick  $i \in I \setminus J, j \in J \setminus I$  such that  $I - i + j$  and  $J - j + i$  are bases. We now continue by induction, where we replace  $J$  by  $J' = J - j + i$ ; note that the size of the symmetric difference has decreased, so we're can induct on the size of the symmetric difference.  $\square$

As we saw, exchanging along all of these perfect matching edges simultaneously need not give us an independent set. However, the following lemma holds.

**Lemma 3** *For all  $I \in \mathcal{I}_1$  and all  $J$  with  $|J| = |I|$ , if the exchange graph  $\mathcal{E}_1^I$  has a unique perfect matching between  $I \setminus J$  and  $J \setminus I$ , then  $J \in \mathcal{I}_1$ .*

**Proof:** Let  $N$  be the unique perfect matching on  $I \Delta J$ . Orient the edges in  $\mathcal{E}_1^I$  so that  $N$  goes  $J \rightarrow I$ , and all other edges go  $I \rightarrow J$ . Suppose there is a directed cycle. Then by swapping edges, we can get another perfect matching and  $N$  wouldn't be unique. So there is no directed cycle, which means that we can order  $I \setminus J$  and  $J \setminus I$  so that the edges in  $N$  go between elements of the same ranking, and all other edges can only go down. In other words, we have  $I \setminus J = \{x_1, \dots, x_t\}$  and  $J \setminus I = \{y_1, \dots, y_t\}$  with the matching edges being  $(y_i, x_i)$  and all other edges going from  $x_i$  to  $y_j$  where  $j > i$ .

Assume that  $J \notin \mathcal{I}_1$ , and pick a circuit  $C \subseteq J$ . Consider  $y_i \in C$  of maximal (“bottom-most”) index  $i$ . Since it’s in  $C$ , we get  $y_i \in \text{span}(C - y_i)$ . If  $x_i$  is the match of  $y_i$  in  $I$ , then there can be no edge from  $x_i$  to an element of  $C$ , since that would be an edge going up. So  $I - x_i + y_j \notin \mathcal{I}_1$  for any  $y_j \in C - y_i$ . Therefore, for any  $y \in C - y_i$ ,  $y \in \text{span}(I - x_i)$ . This implies that  $y_i \in \text{span}(C - y_i) \subseteq \text{span}(I - x_i)$ . Therefore,  $I - x_i + y_i \notin \mathcal{I}_1$ , a contradiction.  $\square$

**Proof of Claim 2.** Let  $P$  be a shortest path from  $X_1$  to  $X_2$  in the directed graph  $\mathcal{E}_1^I \cup \mathcal{E}_2^I$ . We want to show that  $I \triangle V(P) \in \mathcal{I}_1 \cap \mathcal{I}_2$ . We add a new element  $\alpha$  that is independent of everything else in both matroids. We get that

$$\mathcal{E}_1^{I+\alpha} = \mathcal{E}_1^I \cup \{(\alpha, y) : y \in X_1\}, \quad \mathcal{E}_2^{I+\alpha} = \mathcal{E}_2^I \cup \{(y, \alpha) : y \in X_2\}.$$

Denote the starting point of  $P$  by  $y_0 \in X_1$  and the end point by  $y_t \in X_2$ . Consider

$$(P \cap \mathcal{E}_1^I) \cup \{(\alpha, y_0)\}.$$

We claim that this is a unique perfect matching in  $\mathcal{E}_1^{I+\alpha}$  on its vertices. To see this, first suppose there was some other way of matching  $\alpha$ . Then its match would be in  $X_1$ , so starting from that match we’d get a strictly shorter path than  $P$ . So  $\alpha$  has a unique match. For all other vertices, if there were some other matching, then one of its new edges must provide a shortcut on the path  $P$ , contradicting the minimality of  $|P|$ . So by the previous lemma, we know that

$$(I + \alpha) \triangle (V(P) + \alpha) = I \triangle V(P) \in \mathcal{I}_1$$

Arguing analogously shows us that  $I \triangle V(P) \in \mathcal{I}_2$ , which finishes the proof.  $\square$