MATH 233B: Polyhedral techniques in combinatorial optimization
Instructor: Jan Vondrák
Lecture date: 2/16/2017 **Scribe: Yuval Wigderson**

1 Applications of Matroid Intersection

Let's see some combinatorial applications of the matroid intersection min-max formula.

1.1 Bipartite matching

First consider bipartite matching. For a set of edges S, we have that $r_1(S)$ is the number of vertices touched by S in V_1, and similarly for $r_2(S)$. Then the min-max formula gives

$$\max_{M \text{ matching}} |M| = \min_{S \subseteq E} (r_1(S) + r_2(E \setminus S))$$

Notice that the vertices touched by S on the left and by $E \setminus S$ on the right is a vertex cover. Conversely, given any minimal vertex cover, we can define S to be the edges incident with the vertices on the left; by minimality, every vertex in the cover on the right is incident to some edge in $E \setminus S$. Therefore, we get that the right-hand side in the above formula is just the size of the minimal vertex cover, and we get back König's theorem.

1.2 Rainbow spanning trees

Suppose G is connected and its edges are colored by $n-1$ colors. Is it true that there is a spanning tree such that each edge has a different color? Our first matroid, M_1, will just be the graphic matroid for G. M_2 will be the partition matroid defined by the color classes (i.e. independent sets contain at most one edge of each color). Then

$$r_1(S) = n - b_0(S)$$

where $b_0(S)$ is the number of connected components in $G[S]$. Additionally, $r_2(S)$ is the number of colors that S contains. Then the min-max formula is

$$\max_{\substack{F \text{ forest} \\ \text{all edges different colors}}} |F| = \min_{S \subseteq E} (n - b_0(S) + \#(\text{colors in } E \setminus S))$$

Note that on the right-hand side, if $E \setminus S$ intersects some color classes, we might as well force it to take the entire class for each of those colors, since this doesn't change the last term and can only make S smaller and thus only increase $b_0(S)$. So the right-hand side equals

$$\min_{I \subseteq [n-1]} \left(n - b_0 \left(\bigcup_{i \notin I} C_i \right) + |I| \right)$$

where C_i are the color classes and we take $E \setminus S = \bigcup_{i \in I} C_i$. We want to characterize when there is a rainbow spanning tree, which is the same as asking when this minimum is $\geq n-1$. So that happens if and only if, for all $I \subseteq [n-1]$,

$$b_0\left(\bigcup_{i \notin i} C_i\right) \leq |I| + 1$$

A related conjecture of Brualdi-Hollingsworth, which is still open, is that for any proper edge coloing of K_{2n} by $2n-1$ colors ($n \geq 3$), the edges can be decomposed into n rainbow spanning trees. It's known that there exist $\Omega(n/\log n)$ disjoint rainbow spanning trees.

1.3 Rado's theorem

Theorem 1 (Rado) *For a matroid $M = (E, \mathcal{I})$ and a set system $\mathcal{X} = (X_1, \ldots, X_M) \subset 2^E$, there is a transversal in $\mathcal{X}$ independent in M if and only if for all $I \subseteq [m]$,*

$$r_M\left(\bigcup_{i \in I} X_i\right) \geq |I|$$

Proof: The transversal matroid T on E is defined so that $S \subseteq [m]$ is independent if it is a partial transversal. Equivalently, if we define a bipartite graph whose left-hand side is $[m]$ and whose right-hand side is E, and connect $i \in [m]$ to all elements of X_i, then a set $S \subseteq E$ is independent if it can be covered by a matching in this graph. We have $r_T(S)$ to just be the size of the maximal matching in $G([m], S)$. By König's theorem, this equals the minimal vertex cover of $G([m], S)$. This is the same as

$$r_T(S) = \min_{I \subseteq [m]} \left(m - |I| + \left|\bigcup_{i \in I} X_i \cap S\right|\right)$$

So now we want to understand when there is a set of size m that's independent in both M and T. By matroid intersection, this is the same as

$$\min_{S \subseteq E}(r_T(S) + r_M(E \setminus S)) \geq m$$

which by the above is the same as

$$\min_{S \subseteq E}\left(\min_{I \subseteq [m]}\left(m - |I| + \left|\bigcup_{i \in I} X_i \cap S\right|\right) + r_M(E \setminus S)\right) \geq m$$

Notice that if we remove some $x \in \bigcup_{i \in I} X_i \cap S$, then we make the size of that union go down by 1, while $r_m(E \setminus S)$ goes up by at most 1. So for minimizing, we can assume that S does not contain any element of $\bigcup_{i \in I} X_i$. Similarly, if we add to S any element not in $\bigcup_{i \in I}$, then this union does not get larger, while $r_M(E \setminus S)$ can only decrease. So the optimum is to take $S = E \setminus \bigcup_{i \in I} X_i$. Therefore, the condition is true iff

$$\min_{I \subseteq [m]}\left(m - |I| + r_M\left(\bigcup_{i \in I} X_i\right)\right) \geq m$$

Rearranging gives us the desired condition. □

1.4 Common transversals

Now suppose that M is itself the transversal matroid of some set system $\mathcal{Y} = (Y_1, \ldots, Y_m) \subset 2^E$. In this case, we want to find a joint transversal, namely a collection of distinct elements $x_1, \ldots, x_m$ so that $x_i \in X_i$ and $x_i \in Y_{\pi(i)}$ for some permutation $\pi \in S_m$. From the above,

$$r_M(S) = \min_{J \subseteq [m]} \left(m - |J| + \left| \bigcup_{j \in J} Y_j \cap S \right| \right).$$

So by Rado's theorem, a common transversal exists if and only if for all $I \subseteq [m]$,

$$r_M \left(\bigcup_{i \in I} X_i \right) \geq |I|$$

which by the above happens iff for all $I, J \subseteq [m]$,

$$m - |J| + \left| \bigcup_{j \in J} Y_j \cap \bigcup_{i \in I} X_i \right| \geq |I|.$$

Since this is trivial for $|I| + |J| \leq m$, this condition is equivalent to saying that for all $I, J \subseteq [m]$ with $|I| + |J| > m$, we have that

$$\left| \bigcup_{i \in I} X_i \cap \bigcup_{j \in J} Y_j \right| \geq |I| + |J| - m.$$

2 Matroid mapping lemma

Let us prove a useful lemma which will be useful for us next time.

Lemma 2 *For any matroid $M' = (E', \mathcal{I}')$ with rank function r', and any mapping $f : E' \to E$, $M = (E, \mathcal{I})$ is a matroid, where*

$$I \in \mathcal{I} \Longleftrightarrow \exists I' \in \mathcal{I}', I = f(I')$$

The rank function on M is

$$r(S) = \min_{T \subseteq S} (|S \setminus T| + r'(f^{-1}(T))).$$

Proof: $\mathcal{I}$ is closed under taking subsets. For the extension axiom, suppose we have $I, J \in \mathcal{I}$ with $|J| > |I|$. So we can find $I', J' \in \mathcal{I}'$ with $f(I') = I, f(J') = J$, and we might as well assume that $|I'| = |I|, |J'| = |J|$. Additionally, among all such choices, we ensure that $|I' \cap J'|$ is as large as possible. By extension in M', there is some $j' \in J' \setminus I'$ so that $I' + j' \in \mathcal{I}'$. If $f(j') \in J \setminus I$, then we're good, since $I + f(j') \in \mathcal{I}$. If not, then $f(j') \in I \cap J$. So in particular $f(j') \in I$, so there exists some $i' \in I'$ so that $f(i') = f(j')$. Moreover, $i' \notin I' \cap J'$, since elements of the intersection map bijectively on J'. So we've found $i' \in I' \setminus J'$ with $f(i') = f(j')$. But then, we could have picked

$I'' = I' - i' + j'$ as our preimage for I. Moreover, $|I'' \cap J'| > |I' \cap J'|$, contradicting minimality. So we get that this can't happen, and we have extension.

For the rank function, let $S \subseteq E$. We partition $f^{-1}(S)$ as $\coprod_{s \in S} f^{-1}(s)$. The rank of S is the largest independent set in it, and a subset of S is independent if it's the image of an independent set. We might as well choose this preimage to contain at most one element from each of our partitioning blocks, meaning that $I \subseteq S$ is independent if and only if for all $i \in I$ there exist $a_i \in f^{-1}(i)$ so that $\{a_i : i \in I\} \in \mathcal{I}'$. Let P_S be the partition matroid on this partition $\{f^{-1}(s) : s \in S\}$. Then by matroid intersection,

$$\max_{I \subseteq S, I \in \mathcal{I}} |I| = \max_{\substack{I' \in \mathcal{I}' \\ scriptI' \in P_S}} |I'| = \min_{T' \subseteq f^{-1}(S)} (r_{P_S}(T') + r'(f^{-1}(S) \setminus T'))$$

Similarly to before, we might as well consider $T' = \bigcup_{s \in T} f^{-1}(s)$ for some $T \subseteq S$. With this choice, $r_{P_S}(T') = |T|$, and therefore

$$r_M(S) = \min_{T \subseteq S} (|T| + r'(f^{-1}(S \setminus T)))$$

Exchanging the role of T and $S \setminus T$ gives what we wanted. $\square$