MATH 233B: Polyhedral techniques in combinatorial optimization
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1 Matroid Union

Let us discuss another operation on matroids which yield a new matroid. Given matroids $M_i = (E_i, \mathcal{I}_i)$ and $E = \bigcup_{i=1}^k E_i$, we define a set system

$$M_1 \vee M_2 \vee \cdots \vee M_k = (E, \mathcal{I})$$

where

$$\mathcal{I} = \{I_1 \cup \cdots \cup I_k : I_1 \in \mathcal{I}_1, \ldots, I_k \in \mathcal{I}_k\}.$$

Theorem 1 *$M_1 \vee \cdots \vee M_k$ is a matroid, with rank function*

$$r(S) = \min_{T \subseteq S} \left(|S \setminus T| + \sum_{i=1}^k r_{M_i}(T \cap E_i) \right).$$

Proof: If $E_1, \ldots, E_k$ are pairwise disjoint, then we certainly get a matroid (it's easy to check the axioms), and its rank function is

$$r(S) = \sum_{i=1}^k r_{M_i}(S \cap E_i).$$

Now recall the matroid mapping lemma from the previous lecture. We can formally construct the disjoint union, and then map it forward under the map that collapses the disjoint union into the honest one, and the matroid mapping lemma implies that we get a matroid whose rank function is precisely

$$r(S) = \min_{T \subseteq S} \left(|S \setminus T| + \sum_{i=1}^k r_{M_i}(T \cap E_i) \right).$$

□

2 Applications of Matroid Union

Suppose we have some matroid $M = (E, \mathcal{I})$, and we want to know what is the maximum $|S|$ for $S = \bigcup_{i=1}^k I_i$ for $I_i \in \mathcal{I}$. This boils down to asking for $\text{rank}(M \vee M \vee \cdots \vee M)$, where we take k copies of M. By the above formula,

$$\text{rank}(M \vee M \vee \cdots \vee M) = r_{M \vee \cdots \vee M}(E) = \min_{T \subseteq E} (|E \setminus T| + k \cdot r_M(T)).$$

Theorem 2 *M can be partitioned into k independent sets if and only if for every $S \subseteq E$,*

$$r(S) \geq \frac{1}{k}|S|.$$

Proof: We want that $r_{M\vee\cdots\vee M}(E) = |E|$. From the above formula, this is the same as asking that

$$\min_{T\subseteq E}(|E \setminus T| + kr(T)) \geq |E|.$$

This happens if and only if for all $T \subseteq E$,

$$kr(T) \geq |T|$$

□

Theorem 3 *M contains k disjoint bases if and only if for all $S \subseteq E$,*

$$r(E) - r(S) \leq \frac{1}{k}|E \setminus S|.$$

Proof: We have k disjoint bases if and only if

$$r_{M\vee\cdots\vee M}(E) \geq kr_M(E)$$

which happens if and only if for all $S \subseteq E$,

$$|E \setminus S| + kr(S) \geq kr(E).$$

□

A special case of this, which was proved earlier, is the following theorem:

Theorem 4 (Nash-Williams) *Let $G = (V, E)$ be a (multi)graph. It contains k disjoint spanning trees if and only if for every partition $V = V_1 \cup V_2 \cup \cdots \cup V_\ell$, the number of crossing edges is at least $k(\ell - 1)$, namely*

$$\sum_{i<j} e(V_i, V_j) \geq k(\ell - 1).$$

Proof: Let M be the graphic matroid on G. By the previous theorem, M contains k disjoint bases (aka spanning trees) if and only if for all $S \subseteq E$,

$$r(E) - r(S) \leq \frac{1}{k}|E \setminus S|.$$

In a graphic matroid, we know that

$$r(S) = n - b_0(S)$$

where $b_0(S)$ is the number of connected components induced by S. We know that $r(E) = n - 1$, where $|V| = n$. Then the condition becomes

$$b_0(S) - 1 = r(E) - r(S) \leq \frac{1}{k}|E \setminus S|.$$

Without loss of generality, we can include all edges in the components induced by S, since it won't change the left-hand side and will only make the right-hand side smaller. Then letting $V_1, \ldots, V_\ell$

be the components induced by S, this tells us that the number of edges in $E \setminus S$, namely the edges between V_is, should be at least $k(\ell - 1)$, as desired. □

Returning to the question of finding disjoint rainbow spanning trees, we see that this becomes a "matroid intersection-union" question. The question can be asked as follows: given matroids M_1, M_2, does $M_1 \cap M_2$ contain k disjoint common bases? Alternately, can $M_1 \cap M_2$ be partitioned into k common independent sets? We don't know if this question is in P, or whether it's NP-complete, or whether there's a min-max formula.

As an example, consider K_4, and suppose we color the three matchings in three different colors. Let M_1 be the graphic matroid and M_2 the partition matroid induced from this coloring. Then both of these matroids have rank 3, since there are 3 matchings and each spanning tree has size 3. Moreover, $M_1 \cap M_2$ consists of precisely the rainbow spanning trees. Both M_1 and M_2 can be decomposed into two disjoint bases. However, we can't find two disjoint rainbow spanning trees, which shows that the above question is fairly subtle.

3 Integer decomposition property

Theorem 5 *Let $M = (E, I)$ be a matroid. If*

$$x = \left(\frac{a_1}{k}, \ldots, \frac{a_n}{k}\right) \in P(M)$$

where $a_1, \ldots, a_n, k \in \mathbb{Z}$, then

$$x = \frac{1}{k} \sum_{i=1}^{k} \chi_{I_i}.$$

for some $I_i \in \mathcal{I}$. In other words, we know that x is a convex combination of independent sets, but it suffices to take just k of them.

Proof: Replace E by E', where each $e \in E$ is blown up into a_e "parallel copies" of e. More formally, the rank function is defined by

$$r'(S) = r(\pi(S))$$

where $\pi(S) \subseteq E$ is the projection of S to E, namely $\pi(S)$ consists of all $e \in E$ such that S contains at least 1 copy of e. Then for any $S \subseteq E'$, the fact that $x \in P(M)$ tells us that

$$r'(S) = r(\pi(S)) \geq \sum_{e \in \pi(S)} x_e = \sum_{e \in \pi(S)} \frac{a_e}{k} \geq \sum_{e \in E} \frac{|S \cap \pi^{-1}(e)|}{k} = \frac{|S|}{k}.$$

So by one of our earlier theorems, this tells us that E' can be partitioned into k independent sets I'_i (in M'). Since each independent set contains at most one copy of each edge, we get that $I_i = \pi(I'_i)$ is independent in M. Additionally, from the fact that I'_i partition E', we get that the number of times any edge e appears in the I_is is precisely a_e, and thus

$$\sum_{i=1}^{k} \chi_{I_i} = (a_1, \ldots, a_n)$$

which is exactly what we wanted. □

Definition 6 *M is a strongly base-orderable matroid if for any bases B_1, B_2, there is a bijection $\pi : B_1 \to B_2$ if for any $S \subseteq B_1$, $(B_1 \setminus S) \cup \pi(S)$ is again a base. Observe that this is a strengthening of the strong exchange property, which holds for all matroids.*

Examples: Strongly base-orderable matroids include partition matroids, transversal matroids, gammoids, deltoids, but not graphic matroids.

Theorem 7 *For strongly base-orderable matroids M_1, M_2 on a ground set E, we have that $M_1 \cap M_2$ can be partitioned into k common independent sets if and only if both M_1 and M_2 can be.*

Proof: The forward implication is immediate. For the reverse, suppose we can partition

$$E = X_1 \cup \cdots \cup X_k = Y_1 \cup \cdots \cup Y_k$$

such that $X_i \in \mathcal{I}_1, Y_j \in \mathcal{I}_2$. Choose this partition so that $\sum_{i=1}^k |X_i \cap Y_i|$ is maximized. If $\sum_{i=1}^k |X_i \cap Y_i| = |E|$, then we are done, since then they are the same partition. So assume that this doesn't happen. In that case, $X_i \cap Y_j \neq \emptyset$ for some $i \neq j$. Extend X_i to a base C_i of M_1 and Y_i to a base D_i of M_2. Similarly, extend X_j to a base C_j of M_1 and Y_j to a base D_j of M_2.

By the strong base-orderable property, there is a matching on $C_i \triangle C_j$ and a matching on $D_i \triangle D_j$; say these are given by bijection $\pi : C_i \setminus C_j \to C_j \setminus C_i$ and $\sigma : D_i \setminus D_j \to D_j \setminus D_i$. We think of this as a graph on the vertices $C_i \cup C_j \cup D_i \cup D_j$, with edges $(x, \pi(x))$ and $(y, \sigma(y))$; as a graph, it's the union of two matchings. As such, it must be bipartite (since any cycle must have edges from both matchings, and must thus be an even cycle). So we can color the vertices red and blue so that each edge is duochromatic. Applying the swapping property to the red vertices in C_i, we see that the set of all red vertices in $C_i \triangle C_j$, combined with all vertices in $C_i \cap C_j$, is a base in M_1, and analogously for M_2.

Now, replace X_i, X_j by X_i', X_j', which are defined by

$$X_i' = \text{red vertices in } X_i \cup X_j \qquad X_j' = \text{blue vertices in } X_i \cup X_j$$

and similarly for Y_i, Y_j. Then these will still be independent sets by the exchange property, and moreover we get that $X_i' \cap Y_j' = \emptyset = X_j' \cap Y_i'$, since these come from two distinct color classes. So we can increase $\sum |X_i \cap Y_i|$, contradicting our assumption. □