MATH 233B: Polyhedral techniques in combinatorial optimization
Instructor: Jan Vondrák
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1 Lower bound for the matching polytope

This is a recent result by Thomas Rothvoß (2015). Let P be the perfect matching polytope, which we know is defined by

$$P = \{x \geq 0 : \forall v, x(\delta(v)) = 1; \forall U \subseteq V, |U| = \text{odd}, x(\delta(U)) \geq 1\}$$

Its vertices are χ_M for all perfect matchings M. For concreteness, and since it's the hardest, we can work with $G = K_{2n}$. Recall that the slack matrix has rows indexed by constraints and columns indexed by vertices, and $S_{ij} = A_i v_j - b_i$. Since there are only linearly many vertex constraints, we will focus on the odd-set constraints. In our case, the entries are

$$S_{M,U} = |M \cap \delta(U)| - 1$$

for a matching M and an odd set U. Recall the basic result of Yannakakis, namely that the extension complexity of P is just $\text{rank}_+(S)$. A consequence that we used last time is the following lemma:

Lemma 1 $\text{rank}_+(S)$ *is at least the number of valid rectangles necessary to express supp*(S) *as a union.*

Unfortunately, this lemma doesn't work in the case of the perfect matching polytope. For we can consider the following rectangles. Fix two disjoint edges e_1, e_2, and define

$$\mathcal{M}_{e_1,e_2} = \{M \text{ perfect matching} : e_1, e_2 \in M\}$$

and

$$\mathcal{U}_{e_1,e_2} = \{U \subset V : |U| = \text{odd}, e_1, e_2 \in \delta(U)\}$$

Finally, let $\mathcal{R}_{e_1,e_2} = \mathcal{M}_{e_1,e_2} \times \mathcal{U}_{e_1,e_2}$. Then for any $M, U \in \mathcal{R}_{e_1,e_2}$, we have that $|M \cap \delta(U)| \geq 2$, which implies that it's a valid rectangle. Moreover,

$$\bigcup_{\substack{e_1,e_2 \in E \\ \text{disjoint}}} = \text{supp}(S)$$

since any entry in $\text{supp}(S)$ will contain some two edges that are shared (by the definition of S), and thus will be in one such rectangle. So we can cover $\text{supp}(S)$ by $O(n^4)$ rectangles, so we can't get an exponential lower bound in this way.

However, recall that the actual nonnegative rank is precisely the minimal r so that we can write $S = \sum_{i=1}^r u_i v_i^T$ for some nonnegative vectors u_i, v_i. If we try to use this rectangle decomposition, by letting u_i and v_i be the characteristic vectors of $\mathcal{M}_{e_1,e_2}$ and $\mathcal{U}_{e_1,e_2}$, then we will cover some entries in S too many times. Since for some M, U, if $k = |M \cap \delta(U)|$¡ then it is covered by $\binom{k}{2}$ rectangles, namely all choices of e_1, e_2. However, we want to cover it $k - 1$ times, which is the slack, and we can't really reduce this because we have a quadratic gap. In order to prove that this problem is unavoidable, we will use the following.

Theorem 2 (Hyperplane separation bound) *Let $S \in \mathbb{R}_+^{f \times v}$, where v is the number of vertices and f the number of constraints. Let $W \in \mathbb{R}^{f \times v}$, and define*

$$\alpha_W = \max_{I \subseteq [f], J \subseteq [v]} \sum_{i \in I, j \in J} W_{ij}$$

We also consider

$$\|S\|_\infty = \max_{i,j} |S_{ij}|$$

Finally, we consider the Frobenius inner product,

$$\langle W, S \rangle = \sum_{i,j} S_{ij} W_{ij}$$

Then for any W,

$$\text{rank}_+(S) \geq \frac{\langle W, S \rangle}{\|S\|_\infty \alpha_W}$$

Proof: By scaling, we can assume that $\|S\|_\infty = 1$. Note that

$$\alpha_W = \max\{\langle W, R \rangle : R \text{ a rank 1 matrix with } [0,1] \text{ entries}\}$$

This is because we can write $R = uv^T$, and without loss of generality we can rescale them so that $u \in [0,1]^f, v \in [0,1]^v$. Then we can take I, J to be independently random with probabilities of an element being in them determined by the coordinates of u, v. In other words, $\Pr(i \in I) = u_i, \Pr(j \in J) = v_j$. Then

$$R = \mathbb{E}[\chi_I \chi_J^T]$$

Therefore, by linearity,

$$\langle W, R \rangle = \mathbb{E}_{i,j}[\chi_J^T W \chi_I] = \mathbb{E}_{I,J}\left[\sum_{i \in I, j \in J} W_{ij}\right]$$

Now assume that S can be written as $S = \sum_{i=1}^r R_i$, where R_i are nonnegative rank-one matrices. Since we assumed that $\|S\|_\infty = 1$, then each R_i has entries in $[0,1]$. Then

$$\langle W, S \rangle = \sum_{i=1}^r \langle W, R_i \rangle \leq \sum_{i=1}^r \alpha_W = r\alpha_W$$

and thus

$$r \geq \frac{\langle W, S \rangle}{\alpha_W}$$

□

The idea of our lower bound is that we cannot cover the positive entries $S_{M,U}$ by a small number of rectangles without covering some large entries way too many times. In order to make this precise, we need to choose W carefully. We define

$$W_{U,M} = \begin{cases} -\infty & |M \cap \delta(U)| = 1 \qquad \text{(only valid rectangles allowed)} \\ \frac{1}{|Q_3|} & |M \cap \delta(U)| = 3 \\ -\frac{1}{k-1}\frac{1}{|Q_k|} & |M \cap \delta(U)| = k \\ 0 & \text{otherwise} \end{cases}$$

where

$$Q_\ell = \{(U, M) \in \mathcal{U}_{all} \times \mathcal{M}_{all} : |M \cap \delta(U)| = \ell\}$$

and

$$\mathcal{M}_{all} = \{\text{all perfect matchings in } K_n\} \qquad\qquad \mathcal{U}_{all} = \{U \subseteq V : |U| = t\}$$

and we define μ_ℓ to be the uniform measure on Q_ℓ. We write $n = |V| = 3m(k-3) + 2k$, where k is a constant (in fact, $k = 401$). Penultimately, the size of the cut-set we're considering is

$$t = \frac{m+1}{2}(k-3) + 3$$

and we insist that $m = \Theta(n)$ is odd. Finally, the $-\infty$ in W is really some very large negative number that dwarfs everything else in the problem.

We can compute that

$$\begin{aligned}\langle W, S\rangle &= \sum_{(U,M)\in Q_3} \frac{1}{|Q_3|} \cdot 2 - \sum_{(U,M)\in Q_k} \frac{1}{k-1}\frac{1}{|Q_k|}(k-1) \\ &= 2 - 1 = 1\end{aligned}$$

So what we need is an upper bound on α_W.

Lemma 3 (Main Lemma) *For any rectangle $\mathcal{R} = \mathcal{U} \times \mathcal{M}$, where $\mathcal{U} \subseteq \mathcal{U}_{all}, \mathcal{M} \subseteq \mathcal{M}_{all}$,*

$$\sum_{(U,M)\in\mathcal{R}} W_{U,M} \le 2^{-\delta(k)\cdot n}$$

where $\delta(k)$ is some constant depending on k, and $\delta(k) > 0$ for k sufficiently large (in particular, $k = 401$ works).

Assuming this lemma, we get that

$$\text{rank}_+(S) \ge \frac{\langle W, S\rangle}{\|S\|_\infty \alpha_W} \ge \frac{1}{(n/2)2^{-\delta(k)n}} \ge 2^{\Omega(n)}$$

which is what we wanted. So it all boils down to proving the Main Lemma.

Proof of Main Lemma. For a valid rectangle $\mathcal{R}$, we have that

$$\sum_{(U,M)\in\mathcal{R}} W_{U,M} = \mu_3(\mathcal{R}) - \frac{1}{k-1}\mu_k(\mathcal{R})$$

What we want to prove is that

$$\mu_3(\mathcal{R}) \le \frac{400}{k^2}\mu_k(\mathcal{R}) + 2^{-\delta(k)n}$$

If we can show this, then we get that

$$\sum_{(U,M)\in\mathcal{R}} W_{U,M} \le \frac{400}{k^2}\mu_k(\mathcal{R}) + 2^{-\delta(k)n} - \frac{1}{k-1}\mu_k(\mathcal{R})$$

So by taking $k \geq 401$, we get that $400/k^2 - 1/(k-1) \leq 0$, and we get the desired bound.

We use the following notation: a partition T is $(A_1, A_2, \ldots, A_m, C, D, B_1, \ldots, B_m)$ which partition $V = [n]$ so that $|A_i| = k-3, |B_i| = 2(k-3)$, and $|C| = |D| = k$. We will say that a perfect matching M is consistent with T if

$$M \subseteq \bigcup_{i=1}^{M} E(A_i) \cup \bigcup_{i=1}^{m} E(B_i) \cup E(C \cup D)$$

In other words, we allow all edges within blocks, but the only cross edges allowed are between the special blocks C, D. A set U is consistent with T if

$$U = \bigcup_{i \in I} A_i \cup C'$$

where $I \subseteq [m]$, $C' \subseteq C$, and $|U| = t$. By the definition of t, this means that U must consist of either a bunch of blocks and also a 3-element subset of C, or else all of C and one less small block. We denote by $\mathcal{M}_{all}(T)$,$\mathcal{U}_{all}(T)$ the set of all matchings (respectively cuts) consistent with T.

To generate a random pair $(U, M) \sim \mu_k$, we can do the following. First, we pick a uniformly random partition T. Then, we generate a uniformly random perfect matching between C and D, call it F. Finally, let M be a uniformly random extension of F to a perfect matching consistent with T. Similarly, let U be a uniformly random cut-set of size t consistent with T such that $C \subseteq U$. Then by the deinition of consistency, we see that $|M \cap \delta(U)| = k$. Moreover, since we're working in the complete graph, this is a uniformly random pair from μ_k, since everything is invariant under relabeling the vertices.

Similarly to generate $(U, M) \sim \mu_3$, we do the following. As above, we generate T and F uniformly randomly, but then pick a random three edges $H \subset F$. Then extend H randomly to a perfect matching M consistent with T. Similarly, U is a random extension of $V(H) \cap C$ consistent with T. Then again $|M \cap \delta(U)| = 3$, and since everything was uniformly random and invariant under relabelings, this pair is drawn according to μ_3.

For $H \subseteq E(C \cup D)$ a matching and $\mathcal{R} = \mathcal{U} \times \mathcal{M}$ a rectangle, we define

$$P_{\mathcal{M},T}(H) = \Pr_{\substack{M \in \mathcal{M}_{all}(T) \\ \text{uniform}}}[M \in \mathcal{M} \mid H \subseteq M]$$
$$P_{\mathcal{U},T}(H) = \Pr_{\substack{U \in \mathcal{U}_{all}(T) \\ \text{uniform}}}[U \in \mathcal{U} \mid U \cap C = V(H) \cap C]$$

We have one important lemma:

Lemma 4

$$\mu_k(\mathcal{R}) = \mathop{\mathbb{E}}_{T} \mathop{\mathbb{E}}_{\substack{F \subseteq C \times D \\ \textit{perf. matching}}} [P_{\mathcal{M},T}(F) P_{\mathcal{U},T}(F)]$$
$$\mu_3(\mathcal{R}) = \mathop{\mathbb{E}}_{T} \mathop{\mathbb{E}}_{\substack{F \subseteq C \times D \\ \textit{perf. matching}}} \mathop{\mathbb{E}}_{H \in \binom{F}{3}} [P_{\mathcal{M},T}(H) P_{\mathcal{U},T}(H)]$$

This is simply a consequence of the fact we previously stated that these are ways of drawing from μ_3, μ_k. □