Math 233B: Polyhedral techniques in combinatorial optimization
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Lecture date: 1/24/2017 **Original scribe: Michael Kapralov**

1 Polyhedral characterization of non-bipartite matchings

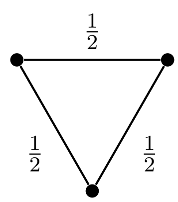
Today, we are interested in the polyhedral description of

$$P_{\text{match}}(G) = \text{conv}\{\chi_M : M \text{ matching in } G\}$$

As we have seen, the description

$$\{\mathbf{x} \geq 0 : \forall v \in V, x(\delta(v)) \leq 1\}$$

is not correct for non-bipartite graphs: For example, a valid constraint missing in this description is



$$\sum_{e \in \text{triangle}} x_e \leq 1$$

since if $\mathbf{x} = \sum_M \alpha_M \chi_M$, then $\sum_{e \in \text{triangle}} x_e = \sum_M \alpha_M |M \cap \text{triangle}| \leq 1$. More generally, for any odd-size set $U \subseteq V$,

$$\sum_{e \in E[U]} x_e \leq \left\lfloor \frac{1}{2}|U| \right\rfloor.$$

We will prove

Theorem 1 *(Edmonds) The matching polytope of G is given by*

$$P_{match}(G) = \left\{ \mathbf{x} \in \mathbb{R}^E : \mathbf{x} \geq 0, \forall v \in V, x(\delta(v)) \leq 1,\ \forall U \subseteq V, |U| = \textit{ odd}, x(E[U]) \leq \left\lfloor \frac{1}{2}|U| \right\rfloor \right\}.$$

Note that the number of constraints is exponential in the size of the graph; however, the description will be still useful for us. But first, let us consider the perfect matching polytope.

2 The perfect matching polytope

Define

$$P_{\text{perf-match}}(G) = \text{conv}\{\chi_M : M \text{ a perfect matching in } G\}$$

In the bipartite case $P_{\text{perf-match}}(G)$ is given by $\{\mathbf{x} \in \mathbb{R}^E : \mathbf{x} \geq 0, \forall v \in V; x(\delta(v)) = 1\}$. Again, we need to add odd-set constraints in the bipartite case.

Theorem 2 *(Edmonds)*

$$\begin{aligned} P_{\textit{perf-match}}(G) \quad = \quad & \{\mathbf{x} \in \mathbb{R}^E : & (1)\\ & \mathbf{x} \geq 0, & (2)\\ & \forall v \in V; x(\delta(v)) = 1, \\ & \forall U \subseteq V, |U| = \textit{ odd}\,; x(\delta(U)) \geq 1\}. \end{aligned}$$

Proof: Let

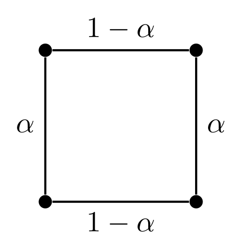
$$Q = \{\mathbf{x} \geq 0 : \forall v \in V, x(\delta(v)) = 1 \text{ and } \forall U \subseteq V \text{ s.t. } |U| \text{ is odd } x(\delta(U)) \geq 1\}.$$

Since every perfect matching in G satisfies the constraints in Q, we have that $P_{\text{perf-match}} \subseteq Q$. We now prove that $Q \subseteq P_{\text{perf-match}}$. Consider a vertex $\mathbf{x} \in Q$. If we prove that $\mathbf{x} \in P_{\text{perf-match}}$ for each vertex of Q, it will imply that $Q \subseteq P_{\text{perf-match}}$, since Q is a bounded polytope and hence a convex hull of its vertices. We proceed by induction on $|E|$.

Base case. $|E| = 1$, we must have $x_e = 1$; trivial.

Inductive step. If $x_e = 0$ for some edge e, we can remove the edge and use the inductive hypothesis. If $x_e = 1$ for some edge e, then we can remove e together with its endpoints and use the inductive hypothesis.

Suppose that $0 < x_e < 1$ for all $e \in E(G)$. Note that the constraints of Q imply that the degree of every vertex is at least 2. First suppose that all degrees are exactly 2. Then E is a union of cycles, and the values x_e on each cycle alternate between $1 - \alpha$ and α for some $\alpha \in (0, 1)$:



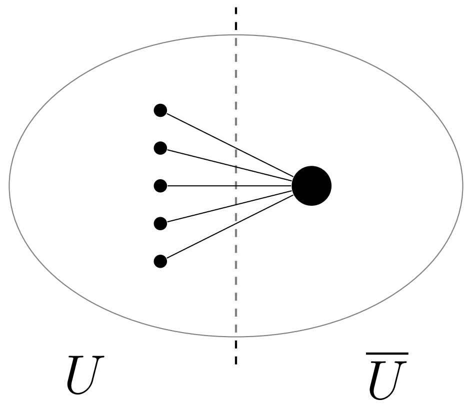
Observe that the cycle must be of even length, since otherwise we would violate the odd-set constraint for it (there are no edges leaving the cycle). This, however, is not a vertex: Consider adding ϵ to every odd edge and subtracting ϵ from every even edge – this can be done for sufficiently small positive and negative ϵ.

Thus, we can assume that some degrees are > 2, and all are ≥ 2. Then we have $|E| > |V|$. Since we are considering a vertex of Q, this implies that there are $|E| > |V|$ linearly independent tight constraints. We have only $|V|$ vertex-degree constraints; hence there exists $U \subseteq V$, $|U|$ odd, such that $x(\delta(U)) = 1$. Let $\overline{U} = V \setminus U$. We must have $|U|, |\overline{U}| \geq 3$, otherwise the constraint is implied by the vertex constraints.

Let G/U be the graph obtained by contracting U to a new node, and $G/\overline{U}$ be the graph obtained by contracting $\overline{U}$ to a new node. (We keep multiple edges in $G/U, G/\overline{U}$ to maintain a one-to-one correspondence between the edges that survived.) Define $\mathbf{x}' \in \mathbb{R}^{E(G/U)}, \mathbf{x}'' \in \mathbb{R}^{E(G/\overline{U})}$ as restrictions of $\mathbf{x}$ to the edges that did not disappear in the contraction.

Claim 3 $\mathbf{x}' \in Q(G/U), \mathbf{x}'' \in Q(G/\overline{U})$.

Proof: Vertex-degree constraints $x(\delta(v)) = 1$ are satisfied because we have $x(\delta(U)) = x(\delta(\overline{U})) = 1$, and the odd set constraints are propagated to to the graph because we shrunk an odd number of vertices into one. □



U was an odd set of size at least 3 and there must be some edges inside U and inside $\overline{U}$, by the tightness of $x(\delta(U)) = 1$. By the inductive hypothesis, $\mathbf{x}' \in P_{\text{perf-match}}(G/U), \mathbf{x}'' \in P_{\text{perf-match}}(G/\overline{U})$.[1] Thus,

$$\mathbf{x}' = \sum_{M' \text{ perf. matching in } G/U} \alpha_{M'} \chi_{M'},$$
$$\mathbf{x}'' = \sum_{M'' \text{ perf. matching in } G/\overline{U}} \alpha_{M''} \chi_{M''}.$$

Note that $\mathbf{x}$ was a vertex of Q, hence a rational point, and consequently $\mathbf{x}', \mathbf{x}''$ are rational as well. By choosing a common denominator, we can write

$$\mathbf{x}' = \frac{1}{N} \sum_{i=1}^{N} \chi_{M_i'},$$
$$\mathbf{x}'' = \frac{1}{N} \sum_{i=1}^{N} \chi_{M_i''},$$

where M_i' (resp. M_i'') are perfeect matchings in G/U (resp. $G/\overline{U}$). Note that by the tightness of $x(\delta(U)) = 1$, each matching here contains exactly one edge in $\delta(U)$. Moreover each particular edge $(u,v) \in \delta(U)$

[1] We do not assume at this point that $\mathbf{x}', \mathbf{x}''$ are *vertices* of the respective polytopes. This will follow at the end of the proof. Is there a way to see it directly?

appears in the same number of matchings among M_i' as among M_i'', because $\mathbf{x}'$ and $\mathbf{x}''$ agree on $\delta(U)$. Therefore, we can pair up the matchings M_i' and M_i'' so that the (unique) edges in $M_i' \cap \delta(U)$ and $M_i'' \cap \delta(U)$ are the same. This implies that $M_i = M_i' \cup M_i''$ is a matching in G. Thus,

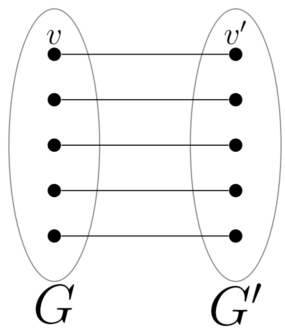
$$\mathbf{x} = \frac{1}{N} \sum_{i=1}^{N} \chi_{M_i}.$$

□

3 The matching polytope

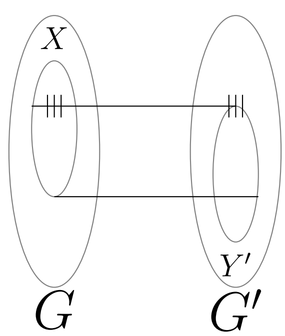
As a consequence, we can derive Theorem 1.

Given G, let G' be a disjoint copy of G. Let $\tilde{G} = G + G' + \bigcup_{v \in V(G)} \{(v, v')\}$, as shown in the figure.



Define $\tilde{\mathbf{x}}$ on the edges of $\tilde{G}$ as $\tilde{x}_e = \tilde{x}_{e'} = x_e$, and $\tilde{x}_{(v,v')} = 1 - x(\delta(v)), \forall v \in V(G)$. If we prove that $\tilde{\mathbf{x}} \in P_{\text{perf-match}}(\tilde{G})$, then we are finished because if $\tilde{\mathbf{x}}$ decomposes into a convex combination of perfect matchings in $\tilde{G}$, by restricting to G we obtain a convex combination of matchings in G that sums up to $\mathbf{x}$.

Claim 4 $\tilde{\mathbf{x}} \in P_{\textit{perf-match}}(\tilde{G})$.



Proof: Denote $\tilde{\delta}(v) = \{\text{edges in } \tilde{G} \text{ incident to } v\}$. First, we have $\tilde{x}(\tilde{\delta}(v)) = 1$ for all $v \in V(\tilde{G})$. We need to prove that $\forall \tilde{U}$ odd in $\tilde{G}$, one has $\tilde{x}(\tilde{\delta}(\tilde{U})) \geq 1$. Consider such $\tilde{U}$ of odd cardinality. Let $\tilde{U} = X \cup Y'$, where $X \subseteq V(G), Y' \subseteq V(G')$ (for any set $S \subseteq V(G)$ we define the corresponding set in G' by S'). Since $|\tilde{U}|$ is odd, we have that either $|X \setminus Y|$ or $|Y \setminus X|$ is odd, wlog assume that $|X \setminus Y|$ is odd.

Using that $\mathbf{x}$ satisfies $x(E[X \setminus Y]) \leq \lfloor \frac{1}{2}|X \setminus Y| \rfloor = \frac{1}{2}|X \setminus Y| - \frac{1}{2}$, we get

$$\tilde{x}(\tilde{\delta}(X \setminus Y)) = \sum_{v \in X \setminus Y} \tilde{x}(\tilde{\delta}(v)) - 2\tilde{x}(E[X \setminus Y]) \geq |X \setminus Y| - 2 \left\lfloor \frac{1}{2}|X \setminus Y| \right\rfloor = 1.$$

Finally,

$$\tilde{x}(\tilde{\delta}(X \cup Y')) \geq \tilde{x}(\tilde{\delta}(X \setminus Y)) \geq 1,$$

where we used the fact that edges that go from $X \setminus Y$ to $X \cap Y$ are counted in $\tilde{x}(\tilde{\delta}(X \setminus Y'))$ as edges from $X' \cap Y'$ to $X' \setminus Y'$ (see the last figure). So $\tilde{\mathbf{x}}$ satisfies the constraints of $P_{\text{perf-match}}(\tilde{G})$. □

4 The Cunningham-Marsh formula

For a given weight vector $\mathbf{w} \in \mathbb{R}^E$, the description of the matching polytope implies that the maximum possible weight of a matching is equal to

$$\max_{\text{matching } M} w(M) = \max\left\{\mathbf{w}^T\mathbf{x} : \mathbf{x} \geq 0, \forall v \in V, x(\delta(v)) \leq 1, \forall |U| = odd, x(E[U]) \leq \lfloor \frac{1}{2}|U| \rfloor\right\}.$$

From LP duality, we obtain

$$\begin{aligned}\max_{\text{matching } M} w(M) \quad = \quad & \min\Big\{ \sum_{v \in V} y_v + \sum_{|U|=odd} \lfloor \frac{1}{2}|U| \rfloor z_U : \\ & \sum_{v \in v} y_v \chi_{\delta(v)} + \sum_{|U|=odd} z_U \chi_{E[U]} \geq \mathbf{w}\Big\}.\end{aligned}$$

Moreover, let us state (without proof) that if $\mathbf{w} \in \mathbb{Z}^E$, then the dual variables can be assumed to be integer as well, and $z_U > 0$ only for sets that form a *laminar family*.[2] This is known as the Cunningham-Marsh min-max formula.

[2] $\mathcal{F}$ is a laminar family, if for any $A, B \in \mathcal{F}$, we have $A \cap B = \emptyset$ or $A \subseteq B$ or $B \subseteq A$.