

1 Spanning Trees

Given a graph $G = (V, E)$ and edge weights $w_e \geq 0$, our goal is to connect all vertices by a subset of edges F while minimizing its cost $\sum_{e \in F} w_e$. Without loss of generality the optimal solution is a tree which is called the Minimum Spanning Tree (MST). This is perhaps the oldest combinatorial optimization problem; it was first solved by Borůvka in 1926 and Jarník in 1930. Both of the proposed algorithms were variants of the greedy algorithm.

Algorithm 1 Borůvka's Algorithm

```
 $F \leftarrow \emptyset$ 
while  $F$  is disconnected do
  for all components  $C_i$  do
     $F \leftarrow F \cup \{e_i\}$  for  $e_i =$  the min-weight edge leaving  $C_i$ .
  end for
end while
```

Algorithm 2 Jarník's Algorithm

```
 $T \leftarrow \emptyset$ 
while  $T$  is not a spanning tree do
   $T \leftarrow T \cup \{e\}$  for  $e =$  the min-weight edge extending the tree  $T$  to a new vertex.
end while
```

In the 1950's the problem was studied again and algorithms were proposed by Prim, Kruskal, and Dijkstra.

Algorithm 3 Kruskal's Algorithm

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 $S \leftarrow E$ 
 $F \leftarrow \emptyset$ 
while  $S \neq \emptyset$  and  $F$  is not spanning do
  Remove the min-weight edge  $e$  from  $S$ .
  if  $F \cup \{e\}$  does not create a cycle then
     $F \leftarrow F \cup \{e\}$ 
  else
    Discard  $e$ 
  end if
end while
```

All of these algorithms work because spanning trees form a “matroid”.

2 Matroids

Definition 1 $\mathcal{M} = (E, \mathcal{I})$, where $\emptyset \neq \mathcal{I} \subseteq 2^E$ is a matroid if

1. $\forall I \in \mathcal{I}, J \subset I \Rightarrow J \in \mathcal{I}$ (“down-closed family”).
2. $\forall I, J \in \mathcal{I}, |I| < |J| \Rightarrow \exists j \in J \setminus I$ such that $I \cup \{j\} \in \mathcal{I}$ (“extension axiom”).

The above definition captures many interesting notions of “independence”, including linear independence among vectors, as we will see below.

2.1 Notation and Terminology

- $I + j \equiv I \cup \{j\}$
- $I - j \equiv I \setminus \{j\}$
- I is an independent set in $\mathcal{M} \iff I \in \mathcal{I}$.
- B is a base in $\mathcal{M} \iff B \in \mathcal{I}$ and B cannot be extended to a larger independent set. More generally, B is a base of $S \subseteq E$, if $B \in \mathcal{I}$ and $\nexists x \in S \setminus B$ such that $B + x \in \mathcal{I}$.

Lemma 2 $\mathcal{M} = (E, \mathcal{I})$ is a matroid $\iff \mathcal{I}$ is down-closed and $\forall S \subseteq E$ all bases of S have the same size.

Proof:

1. If \mathcal{M} is a matroid and B_1, B_2 are bases of $S \subseteq E$, then if $|B_1| < |B_2| \implies B_1$ can be extended by $x \in B_2 \setminus B_1$, hence B_1 was not a base.
2. Assume that $\forall S \subseteq E$ all bases have the same size. Then for $I, J \subseteq E$ where $|I| < |J|$ let $S = I \cup J$. Then I is not a base of S , therefore $\exists j \in S \setminus I = J \setminus I$ such that $I + j \in \mathcal{I}$.

□

2.2 Graphic Matroids

So why are spanning trees an example of a matroid?

Definition 3 For a graph $G = (V, E)$ a forest is any set of edges $F \subseteq E$ that does not contain any cycles.

Lemma 4 $\mathcal{M} = (E, \mathcal{F})$ where $\mathcal{F} = \{F \subseteq E : F \text{ is a forest}\}$ is a matroid.

Proof: \mathcal{F} is clearly down-closed. We prove that for any $S \subseteq E$, all bases have the same size. Take any $S \subseteq E$ and look at the connected components of (V, S) . What are the bases of S , i.e. maximal subsets $F \subseteq S$ which are forests? In each connected component C_i we have $|F \cap E[C_i]| = |C_i| - 1$, because if we have less than $|C_i| - 1$ edges, then $F \cap E[C_i]$ is not connected on C_i and we can add some edge without creating a cycle. Conversely, we cannot have more than $|C_i| - 1$ edges on C_i because we would create a cycle. This implies that $|F| = \sum |C_i| - 1 = |V| - \#\text{components}(S)$, the same number for each base F . □

Definition 5 A matroid $\mathcal{M} = (E, \mathcal{I})$ is graphic if $\mathcal{I} = \{F \subseteq E : F \text{ is acyclic}\}$ for some graph $G = (V, E)$.

2.3 The Rank Function

Definition 6 The rank function of a matroid $\mathcal{M} = (E, \mathcal{I})$ is $r(S) = \max\{|I| : I \in \mathcal{I}, I \subseteq S\}$, for any $S \subseteq E$.

Recall that all bases of S have the same size, i.e. the rank function says what this size is. This is identical to the notion of dimension in vector spaces. We already proved the following.

Lemma 7 In graphic matroids, $r(S) = |V| - \#\text{components}(S)$.

2.4 Other Examples of Matroids

1. Linear Matroid: $\mathcal{M} = (E, \mathcal{I})$ where $E \subseteq \mathbb{V}$ for some vector space \mathbb{V} and $I \in \mathcal{I} \iff I$ is linearly independent in \mathbb{V} .
2. Uniform Matroid: $\mathcal{I} = \{I : |I| \leq k\}$.
3. Partition Matroid: $E = E_1 \cup E_2 \cup \dots \cup E_\ell$, a disjoint union, and $k_1, k_2, \dots, k_\ell \in \mathbb{Z}$ such that $\mathcal{I} = \{I \subseteq E : \forall j; |I \cap E_j| \leq k_j\}$.
4. Transversal Matroid: Given $X_1, X_2, \dots, X_\ell \subseteq E$, not necessarily disjoint,

$$\begin{aligned}\mathcal{I} &= \{T \subseteq E : T \text{ is a partial transversal of } X_1, X_2, \dots, X_\ell\} \\ &= \{T \subseteq E : T = \{t_1, \dots, t_k\}, t_j \in X_{i_j} \text{ where } i_1, \dots, i_k \text{ are distinct}\}\end{aligned}$$

5. Matching Matroid: (generalizes transversal matroids) Given a graph $G = (V, E)$, let $\mathcal{M} = (V, \mathcal{I})$ where $\mathcal{I} = \{I \subseteq V : I \text{ is covered by some matching in } G\}$.
6. Gammoid: (generalizes transversal matroids) Given a directed graph $D = (V, A)$ and two sets of vertices $S, T \subseteq V$, let $\mathcal{M} = (T, \mathcal{I})$ where $\mathcal{I} = \{J \subseteq T : \exists I \subseteq S, |I| = |J|, \text{ and there are } |I| \text{ vertex-disjoint paths from } I \text{ to } J\}$.
7. Deltoid: (special case of a transversal matroid) Given a bipartite graph $G = (V_1 \cup V_2, E)$, let \mathcal{M} be the transversal matroid defined by the set system $X_v = \{v\} \cup N(v)$ for each $v \in V_1$.

Equivalently, one can define the bases of \mathcal{M} as $B = V_1 \Delta A$ where A are the vertices of some matching in G .

All the matroids above are special cases of linear matroids, i.e. can be represented by linear independence among vectors in some vector space.