Math 233B: Polyhedral techniques in combinatorial optimization
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Lecture date: 2/2/2017 **Original scribe: Shayan Oveis Gharan**

1 The greedy algorithm for matroids

The following algorithm finds the maximum weight base in a matroid $\mathcal{M} = (E, \mathcal{I})$

Algorithm 1 Greedy algorithm for selecting the max-weight base of a matroid

Input: a matroid $\mathcal{M} = (E, \mathcal{I})$, where $E = \{1, 2, \ldots, n\}$ is the ground set, and weight of i is w_i.
Output: A base $B \in \mathcal{I}$ such that $w(B) = \max_{B \in \mathcal{B}} w(B)$.

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Relabel the elements of the matroid so that $w_1 \geq w_2 \geq \ldots \geq w_n$.
$S \leftarrow \emptyset$.
for $i \leftarrow 1$ to $n$ do
    if $S + i \in \mathcal{I}$ then
        $S \leftarrow S + i$.
    end if
end for
return $S$
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Theorem 1 (Rado/Gale) *For any ground set $E = \{1, 2, \ldots, n\}$, and a family of subsets $\mathcal{I} \subset 2^E$, Algorithm 1 returns the maximum-weight base for any set of weights $w : E \rightarrow \mathbb{R}$ if and only if $\mathcal{M} = (E, \mathcal{I})$ is a matroid.*

We prove this theorem in two parts.

Claim 2 *($\Leftarrow$ part) Suppose that $(E, \mathcal{I})$ is a matroid. For any set of weights assigned to the elements of E, Algorithm 1 returns the maximum-weight base.*

Proof: Wlog assume that $w_1 \geq w_2 \geq \ldots \geq w_n$. We prove that at any point of the execution of the algorithm, there exists an optimal base B such that $S \subseteq B$ and $B \setminus S$ is among the remaining elements.

In particular, let S_i be the set S after observing the first i elements (e.g. $S_0 = \emptyset$). We use induction to show that for any S_i, there exists an optimal base B_i such that $S \subseteq B_i$, and $B_i \setminus S_i \subset \{i+1, \ldots, n\}$. Note this certainly implies the claim, since we get $S_n = B_n$ is an optimal base. The base case of the induction is trivially satisfied for $S_0 = \emptyset$.

Suppose before the i-th iteration we have $S_{i-1} \subset B_{i-1}$ where B_{i-1} is a max-weight base. If the algorithm does not add i to S_{i-1}, it means that $S_{i-1} + i \notin \mathcal{I}$ and therefore i cannot be in B_{i-1} (by the downward closed-ness property of the matroid). Therefore, $B_i = B_{i-1}$ satisfies the induction statement.

Now assume that the algorithm adds i to S_{i-1}. By the induction hypothesis, if $i \in B_{i-1}$, then $S_i = S_{i-1} + i \subset B_{i-1}$, and we can set $B_i = B_{i-1}$. Otherwise, by the extension axiom of the matroid $\mathcal{M}$, the set S_i can be extended from B_{i-1} until it becomes a base, say B'. Since $i \notin B_{i-1}$, we have

$B' \setminus B_{i-1} = \{i\}$. Moreover, since B' is also a base, we have $|B_{i-1} \setminus B'| = 1$, so let $B_{i-1} \setminus B' = \{j\}$. Therefore we can write $B' = B_{i-1} + i - j$.

Since $B_{i-1} \setminus S_i \subset \{i+1, \ldots, n\}$, we have $j \in \{i+1, \ldots, n\}$ is one of the remaining elements. Therefore, since the algorithm orders the element decreasingly according to their weights, we have $w_j \leq w_i$. But this means that $w(B') = w(B_{i-1}) + w_i - w_j \geq w(B_{i-1})$. By the optimality assumption of B_{i-1}, we have $w(B') = w(B_{i-1})$, hence $B_i = B'$ satisfies the induction statement. □

Claim 3 *($\Rightarrow$ part) Suppose $(E, \mathcal{I})$ is not a matroid. There exists an assignment of weights to the elements of E such that algorithm 1 does not return a maximum-weight base.*

Proof: If $(E, \mathcal{I})$ is not a matroid, it does not satisfy at least one of the two properties of the matroid. Suppose $\mathcal{I}$ is not a downward-closed family of sets. Therefore, there exist two sets $S \subset T$, $T \in \mathcal{I}$, but $S \notin \mathcal{I}$. Suppose we assign the weights as follows:

$$\forall 1 \leq i \leq n, \; w_i = \begin{cases} 2 & i \in S \\ 1 & i \in T \setminus S \\ 0 & \text{otherwise} \end{cases}$$

By the weight assignment, the algorithm first considers the elements of S, then the elements of T, and then the rest of the elements. The elements in $E \setminus S$ are worth nothing, thus every optimal base must contain T. Suppose the algorithm selects a subset $S_1 \subset S$ after observing the elements of S. Since $S \notin \mathcal{I}$, we have $S_1 \neq S$. Out of the remaining elements, the algorithm can get value at most $|T \setminus S|$. If S_2 is the final set chosen by the algorithm, we have

$$w(S_2) = 2|S_1| + w(S_2 \setminus S) < 2|S| + |T \setminus S| = w(T).$$

Now suppose $(E, \mathcal{I})$ is not a matroid because the extension axiom is violated (assume the downward closed property). In particular, let $S, T \in \mathcal{I}$ be two independent sets such that $|S| < |T|$, and for all $i \in T \setminus S$, $S + i \notin \mathcal{I}$. We use the following weights:

$$\forall 1 \leq i \leq n, \; w_i = \begin{cases} 1 + \frac{1}{2|S|} & i \in S \\ 1 & i \in T \setminus S \\ 0 & \text{otherwise} \end{cases}$$

Note that S is not necessarily a subset of T here. This time, because of the downward closedness property the algorithm would select all of the elements of S. But this means that it can not add any element in $T \setminus S$, as this would violate independence. Further elements do not bring any value anymore, so if S_2 is the solution returned by the algorithm,

$$w(S_2) = w(S) = |S| \left(1 + \frac{1}{2|S|}\right) = |S| + \frac{1}{2}$$

while the value of T is

$$w(T) \geq |T| \geq |S| + 1.$$

□

The following properties can be shown using the above theorem.

1. Let S_i be the set of elements chosen by the algorithm after observing the first i elements. Then S_i is always a base of those i elements. (By considering $w_1 = \ldots = w_i = 1$ and $w_{i+1} = \ldots = w_n = 0$.)

2. Finding the maximum-weight base in a matroid is in fact equivalent to finding the minimum-weight base. Let $w_{max} = \max_{1 \le i \le n} w_i$ be the maximum weight assigned to the elements; to find the minimum-weight base it is sufficient to consider $w'_i = w_{max} - w_i$, for all $i \in E$.

3. Also, it is straightforward that if the weights are non-negative, then the maximum-weight independent set is the same as the maximum-weight base. In general, we can say that the maximum-weight independent set is the maximum-weight base of the elements with non-negative weights.

2 The span function in matroids

The following definition of a "span" of a set of elements in a matroid is a generalization of the notion of span in vector spaces.

Definition 4 *Let $\mathcal{M} = (E, \mathcal{I})$ be a matroid. For any set $S \subseteq E$ define*

$$span(S) := \{i \in E : \ rank(S+i) = rank(S)\}.$$

Lemma 5 *Let $S \subseteq E$. Then, any base of S is also a base of $span(S)$.*

Proof: By contradiction: Let B be a base of S which is not a base of span(S). Since B is a base of S, we have $|B| = \text{rank}(S)$. Also, since B is not a base of span(S), it means that there is some element $i \in \text{span}(S) \setminus B$, such that $B + i \in \mathcal{I}$. Therefore,

$$\text{rank}(S+i) \ge \text{rank}(B+i) > \text{rank}(B) = \text{rank}(S).$$

This contradicts the definition of $i \in \text{span}(S)$. □

Lemma 6 *Let $S \subseteq E$. For any base B of S and any element $i \in E \setminus S$, $i \in span(S)$ if and only if $B + i \notin \mathcal{I}$.*

Proof: If $i \in span(S)$ and B is a base of S, then by Lemma 5, B is also a base of $span(S)$, and thus a base of $S + i$. Therefore, $B + i \notin \mathcal{I}$.

Conversely, suppose $i \notin span(S)$. Therefore, $rank(S+i) > rank(S)$, and there is an independent set $B' + i$, where $B' \subseteq S$ and $|B' + i| = rank(S) + 1$. In other words, B' is a base of S. Now consider any base B of S. This is also an independent subset of $S + i$. Since $|B| < |B' + i|$, by the extension axiom, it can be extended by adding an element from $B' + i$. But that element must be i (otherwise, B was not a base of S), and thus $B + i$ is independent. □

Next, we prove that span preserves the ordering by inclusion.

Lemma 7 *For any $S \subseteq T \subseteq E$, $span(S) \subseteq span(T)$.*

Proof: Let B_S be a base of S, and B_T a base of T. By the extension axiom, B_S can be extended to a base B' of T from the elements of B_T (note that $B' \setminus B_S \subseteq T \setminus S$).

Consider $i \in span(S)$. Since $rank(S + i) = rank(S)$, we have $B_S + i \notin \mathcal{I}$. Therefore, since $B_S \subseteq B'$, by the downward closedness axiom, $B' + i \notin \mathcal{I}$ either. By Lemma 6, $i \in span(T)$. $\square$

Suppose we assign distinct weight to the elements of the matroid (i.e. $w_i \neq w_j$ for all $i, j \in E$), then the maximum weight base is unique. Using the facts above, we can describe the maximum-weight base as follows:

Lemma 8 *Let $\mathcal{M} = (E, \mathcal{I})$ be a matroid, $E = \{1, 2, \ldots, n\}$ and assume $w_1 > w_2 > \ldots > w_n$. Then, the maximum-weight base is*

$$B_{opt} = \{i \in E : i \notin span(\{1, \ldots, i-1\})\}.$$

Proof: Consider Algorithm 1. Let $E_i = \{1, 2, \ldots, i\}$ be the set of the first i elements observed by the algorithm, and similar to the proof of Theorem 1, let S_{i-1} be the independent set chosen by the algorithm after observing the elements of E_{i-1}. Recall that S_{i-1} is a base of E_{i-1}. Therefore, by Lemma 6, $S_{i-1} + i \in \mathcal{I}$ if and only if $i \notin span(S_{i-1}) = span(E_{i-1})$. So the algorithm produces exactly the set B_{opt}. From the analysis of Algorithm 1, it is also clear that in this case the maximum-weight base in unique. $\square$

3 Characterization of rank functions

In this section we prove some of the basic properties of rank functions.

Lemma 9 *The rank function of a matroid satisfies the following:*

1. *For any $S \subseteq T \subseteq E$ of elements, we have $r(S) \leq r(T)$ (**monotonicity**)*
2. *For any $S \subseteq T \subseteq E$, $i \in E \setminus T$, we have $r(T + i) - r(T) \leq r(S + i) - r(S)$ (**non-increasing marginal values**)*

Proof: The first property is trivial (since any base of S is also an independent set of T). To prove the second property we use Lemma 7: Observe that $r(S + i) - r(S) = 0$ if $i \in span(S)$ and 1 if $i \notin span(S)$. Lemma 7 implies that if $i \in span(S)$ then $i \in span(T)$. Therefore,

$$r(T + i) - r(T) \leq r(S + i) - r(S).$$

$\square$

For a set S and an element $i \notin S$, we call $r(S + i) - r(S)$ the marginal value of i with respect to S. In the next lemma we show that non-increasing marginal values are equivalent to *submodularity*: a function f is submodular if $f(A \cup B) + f(A \cap B) \leq f(A) + f(B)$ for every pair of sets A, B.

Lemma 10 *Let $f : 2^E \to \mathbb{R}$ be a set function on a ground set E. Then f is submodular ($\forall A, B \subseteq E$, $f(A \cap B) + f(A \cup B) \leq f(A) + f(B)$) if and only if for all $S \subset T \subset E$ and $i \in E \setminus T$:*

$$f(T + i) - f(T) \leq f(S + i) - f(S).$$

Proof: Assume for all $S \subset T$ and $i \notin T$, we have $f(T+i) - f(T) \leq f(S+i) - f(S)$. Let $A, B \subseteq E$ be two subsets of E. If $B \subseteq A$, the claim is trivial. Let $B \setminus A = \{b_1, b_2, \ldots, b_k\}$. We have:

$$\begin{aligned} f(A \cup B) - f(A) &= \sum_{i=1}^{k} (f(A + b_1 + \ldots + b_i) - f(A + b_1 + \ldots + b_{i-1})) \\ &\leq \sum_{i=1}^{k} (f(A \cap B + b_1 + \ldots + b_i) - f(A \cap B + b_1 + \ldots + b_{i-1})) \\ &= f(B) - f(A \cap B). \end{aligned}$$

Here the inequality follows from the assumption once we set $S := A \cap B$ and $T := A$ (note that this implies $S \subseteq T$).

Conversely, suppose for any two sets A, B we have $f(A \cup B) + f(A \cap B) \leq f(A) + f(B)$. Let $S \subset T$ and $i \notin T$. Now set $A := S + i$ and $B := T$. By the submodularity condition, we get:

$$f(T+i) + f(S) = f(A \cup B) + f(A \cap B) \leq f(A) + f(B) = f(S+i) + f(T),$$

which completes the proof. □

It turns out that submodularity, together with the fact that marginal values are 0 or 1, characterizes exactly the rank functions of matroids.

Lemma 11 *A function $r : 2^E \mapsto \mathbb{R}$ is a rank function of a matroid if and only if*

1. $r(\emptyset) = 0$ *and* $r(S+i) - r(S) \in \{0, 1\}$ *for all* $S \subseteq E, i \notin S$;
2. r *is submodular, i.e.* $r(S \cup T) + r(S \cap T) \leq r(S) + r(T)$ *for all* $S, T \subseteq E$.

Proof: We already know that any matroid rank function must satisfy these conditions. So let us assume that r satisfies the conditions and define $\mathcal{I} = \{A : r(A) = |A|\}$. We claim that $(E, \mathcal{I})$ is a matroid and r is its rank function.

By the first condition on r, is it clear that $\mathcal{I}$ is closed under taking subsets. We claim that for any set S, all maximal subsets of S which are in $\mathcal{I}$ (bases of S) have the same size. Consider any $A \subseteq S$ such that $r(A) = |A|$. We have $r(A+i) - r(A) \in \{0, 1\}$. As long as $r(A) < r(S)$, by submodularity there is $i \in S \setminus A$ such that $r(A+i) = r(A) + 1$. Hence, if A is maximal such that $r(A) = |A|$, we have $r(A) = r(S)$. i.e., all bases have the same size, and $r(S)$ is equal to that size. □