Approximability of Multiway Partitioning Problems and Lower Bounds from Sperner's Colorings

Jan Vondrák[1]

[1] IBM Almaden Research Center
San Jose, CA

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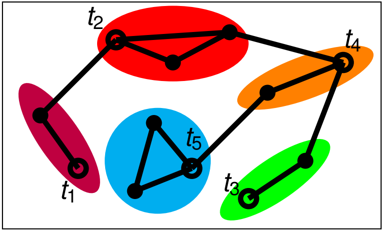
Problem 1: Multiway Cut

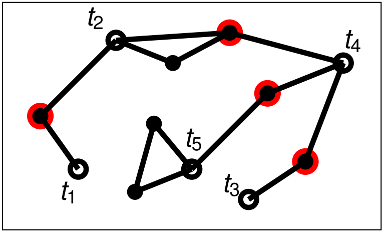
Input: Graph G, terminals $t_1, \ldots, t_k \in V$, edge weights $w(e)$.



Output: $V = S_1 \cup S_2 \cup \ldots \cup S_k$ so that $t_i \in S_i \ \forall i \in [k]$.
Objective: Minimize the weight of cut edges, $\sum_{i \neq j} \sum_{e \in E(S_i, S_j)} w(e)$.

Problem 2: Node-weighted Multiway Cut

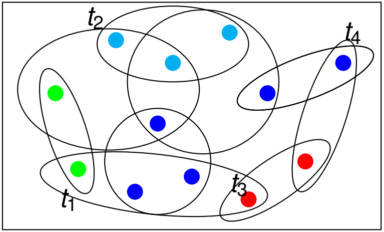
Input: Graph G, terminals $t_1, \ldots, t_k \in V$, node weights $w(v)$.



Output: $W \subset V$ such that the terminals are separated in $G[V \setminus W]$.
Objective: Minimize the weight of the removed vertices $\sum_{v \in W} w(v)$.

Problem 3: Hypergraph Multiway Cut

Input: Hypergraph $H = (V, E)$, terminals $t_1, \ldots, t_k \in V$, hyperedge weights $w(e)$.



Output: Partition $V = S_1 \cup S_2 \cup \ldots \cup S_k$ such that $t_i \in S_i$ $\forall i \in [k]$.
Objective: Minimize the weight of cut hyperedges, $\sum_{e \in E, e \not\subseteq S_i} w(e)$.

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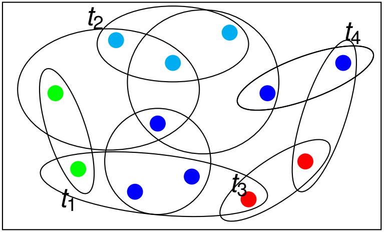

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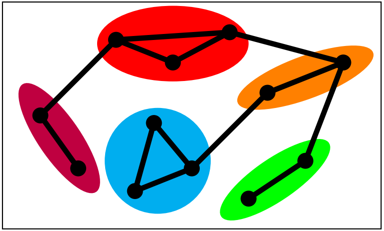
...approximation-equivalent to Node-weighted Multiway Cut!

Problem 4: (Uniform) Metric Labeling

Input: Graph $G = (V, E)$, assignment costs $c : V \times [k] \to \mathbb{R}_+$, metric $d : [k] \times [k] \to \mathbb{R}_+$.



Output: Labeling $\ell : V \to [k]$.
Objective: Minimize assignment+separation cost,

$$\sum_{v \in V} c(v, \ell(v)) + \sum_{(u,v) \in E} d(\ell(u), \ell(v)).$$

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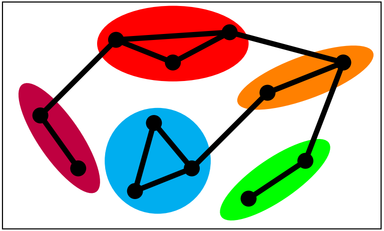

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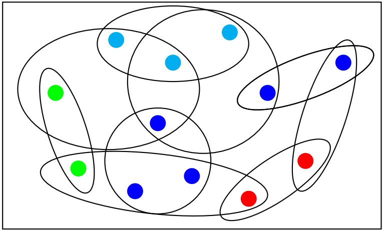
...generalizes Multiway Cut, even with the uniform metric $d(u, v) = \delta_{uv}$.

Problem 5: Hypergraph Labeling

Input: Hypergraph $H = (V, E)$, assignment costs $c : V \times [k] \to \mathbb{R}_+$, hyperedge weights $w(e)$.



Output: Labeling $\ell : V \to [k]$.
Objective: Minimize the total assignment + separation cost:

$$\sum_{v \in V} c(v, \ell(v)) + \sum_{e \in E, |\ell[e]| > 1} w(e).$$

Min-CSP formulation

What do these problems have in common?

Min-CSP form:
ground set V, label set $[k]$, predicates $\Psi_e : [k]^e \to [0,1]$ for $e \in E$.

Goal: Find a labeling $\ell : V \to [k]$ minimizing

$$\sum_{e=(v_1,\ldots,v_j)\in E} w_e \Psi_e(\ell(v_1),\ldots,\ell(v_j)).$$

In particular, our problems involve the Not-Equal predicate:
$\Psi(\ell_1,\ell_2) = 1$ if $\ell_1 \neq \ell_2$ and 0 otherwise.

Local Distribution LP for Min-CSP [Ene-V.-Wu '13]

- [Calinescu-Karloff-Rabani '98] Geometric relaxation for Multiway Cut: *embed vertices in a simplex and minimize* $\sum_{(u,v)\in E} \|\mathbf{x}_u - \mathbf{x}_v\|_1$.
- [Chekuri-Khanna-Naor-Zosin '05] Earthmover LP for Metric Labeling.

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Local Distribution LP:
Variables $x_{v,i} (v \in V, i \in [k])$ and $y_{e,\alpha} (e \in E, \alpha \in [k]^e)$.

$$\begin{aligned}
\text{Minimize} \quad & \textstyle\sum_{e\in E} \sum_{\alpha\in[k]^e} y_{e,\alpha} \Psi_e(\alpha) : \\
\forall v \in e \in E; \quad & \textstyle\sum_{\alpha\in[k]^e : \alpha(v)=\ell} y_{e,\alpha} = x_{v,\ell}, \\
\forall v \in V; \quad & \textstyle\sum_{i=1}^{k} x_{v,i} = 1, \\
& x_{v,i}, y_{e,\alpha} \ge 0.
\end{aligned}$$

Optimality of the Local Distribution LP

Theorem

For any Min-CSP problem with a collection of predicates including Not-Equal, if there is an instance of the Local Distribution LP with integrality gap γ, then it is Unique-Games-hard to achieve a $(\gamma - \epsilon)$-approximation for any fixed $\epsilon > 0$.

[Ene-V.-Wu '13], building on [Manokaran,Naor,Raghavendra,Schwartz '08]

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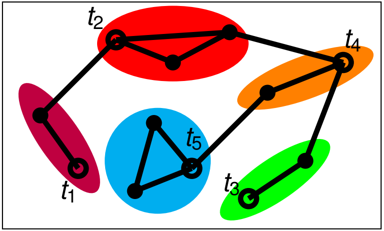
Question*: Is there a rounding algorithm that matches the integrality gap of the Local Distribution LP for any Min-CSP?*

Submodular generalizations

Multiway cut problems have natural “submodular generalizations".

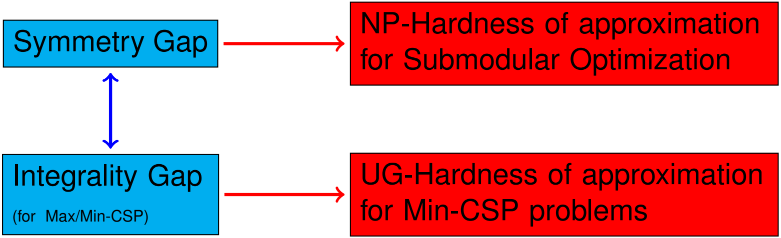
E.g.: *Multiway Cut* $\longrightarrow$ *Submodular Multiway Cut*.
Input: Submodular function $f : 2^V \to \mathbb{R}_+$, terminals $t_1, \ldots, t_k \in V$.



Output: Partition $V = S_1 \cup S_2 \cup \ldots \cup S_k$, $\forall i \in [k]; t_i \in S_i$.
Objective: Minimize $\sum_{i=1}^k f(S_i)$.

Integrality gap vs. Symmetry gap

- Symmetry gap is the gap between “symmetric" and “asymmetric" solutions, in the “multilinear relaxation".
- It implies hardness for submodular optimization: in the oracle model [V. ’09] and NP-hardness for explicit instances [Dobzinski,V. ’12].
- It happens to be equal to the integrality gap of the Local Distribution LP for a related instance.



What is known for concrete problems

Multiway Cut: *not completely understood*
Approximation: 1.2965 [Sharma-V. '14]
UG-hardness: $8/7 - \epsilon \simeq 1.1428$ [Freund-Karloff '00 + MNRS '08]

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Node-weighted / Hypergraph / Submodular Multiway Cut: *resolved*
Approximation: $2 - 2/k$ [Garg-Vazirani-Yannakakis '94, Chekuri-Ene '11, EVW '13]
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Metric Labeling: *resolved (mod UGC)*
Approximation / UG-hardness: $\Theta(\log k)$ [Kleinberg-Tardos '98 + FRT '04] / [MNRS '08]

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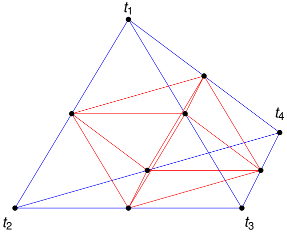
Hypergraph Labeling: *resolved (mod UGC)*
Approximation / UG-hardness: $k - 1$ [Chekuri-Ene '11, Ene-V. '14, Mirzakhani-V. '14]

Integrality gap for Multiway Cut

$\frac{8}{7+\frac{1}{k-1}}$ integrality gap for Multiway Cut [Freund-Karloff '00]

Blue weight = 1, Red weight = $\frac{3}{2k}$.



Two optimal assignments:

- All internal nodes to one terminal
- Internal node $\frac{1}{2}(t_i + t_j)$ to terminal $t_{\min(i,j)}$.

Balancing them gives $\frac{8}{7+\frac{1}{k-1}} LP$.

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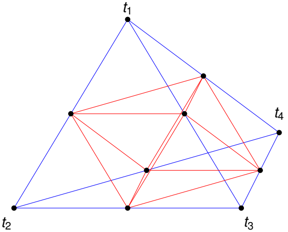

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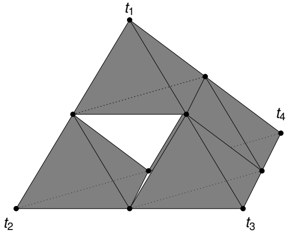
Rounding? quite far away: 1.2965 [Sharma-V. '14]
(combination of several randomized schemes)

Integrality gap for Hypergraph Multiway Cut

$2 - 2/k$ integrality gap for Hypergraph Multiway Cut
($\simeq$ Node-weighted Multiway Cut)

hyperedges = gray simplices



- LP cost $= k \times \frac{1}{2} = \frac{k}{2}$.
- Every assignment cuts at least $k - 1$ hyperedges.

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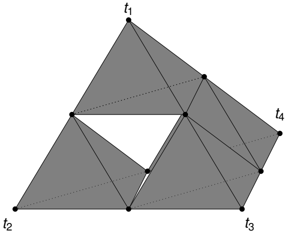

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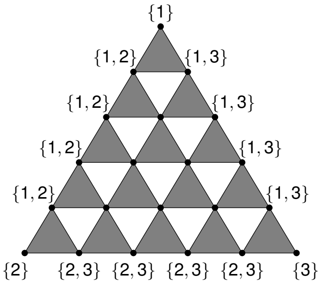
- Every assignment cuts at least $k - 1$ hyperedges.

Matching rounding: (even for Submodular Multiway Cut) [Chekuri-Ene '11]
Draw a random threshold $\theta \in [\frac{1}{2}, 1]$, allocate $\{v : x_{v,i} > \theta\}$ to terminal i, allocate the rest to a random terminal.

Integrality Gap for Hypergraph Labeling

Problem: Color lists $L(v), v \in V$; find a labeling $\ell(v) \in L(v)$ that minimizes # non-monochromatic hyperedges.

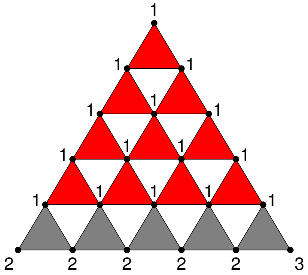
hyperedges = gray simplices
color lists = as in Sperner's Lemma



Question: [Ene-V. '14]
What is the minimum possible number of *non-monochromatic hyperedges*?

Optimal Coloring => Integrality gap

Plausible guess: [Ene-V. '14] the following "simple coloring" has the minimum number of non-monochromatic hyperedges:



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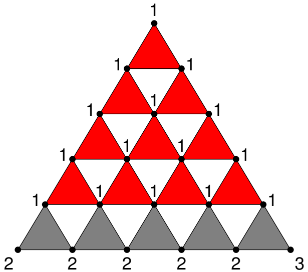
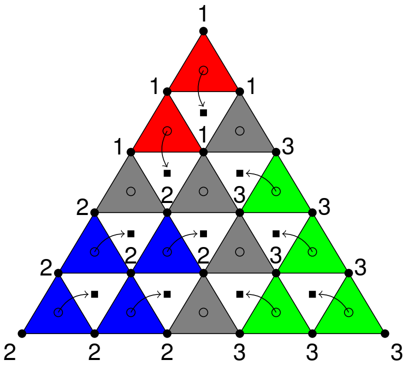


This gives an integrality gap:

- Total # hyperedges $\simeq \frac{1}{(k-1)!} n^k$
- LP cost $\simeq \frac{1}{n} \cdot \frac{1}{(k-1)!} n^k = \frac{1}{(k-1)!} n^{k-1}$
- Optimum cost $\simeq \frac{1}{(k-2)!} n^{k-1} = (k-1) \times \mathit{LP\ cost}$

Proof of the optimality of the "simple coloring"



[Mirzakhani-V. '14]

- Set of vertices $= V_{n,k}$, hyperedges can be identified with $V_{n-1,k}$
- Map *monochromatic hyperedges* to $V_{n-2,k}$ as in the picture.

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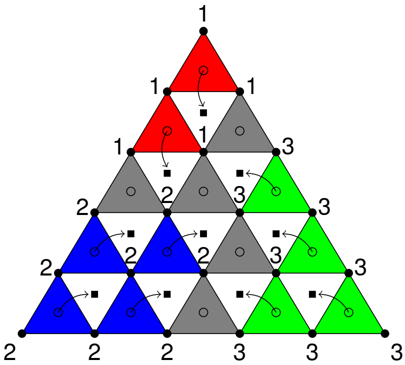
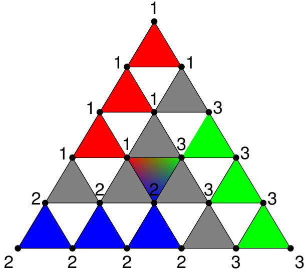


- Set of vertices $= V_{n,k}$, hyperedges can be identified with $V_{n-1,k}$
- Map *monochromatic hyperedges* to $V_{n-2,k}$ as in the picture.
- Since two monochromatic hyperedges cannot be neighbors, the mapping is injective.
- $\#\text{non-mono}\chi \geq |V_{n-1,k}| - |V_{n-2,k}| = \binom{n+k-2}{k-1} - \binom{n+k-3}{k-1} = \binom{n+k-3}{k-2}$.

What about Sperner's Lemma?

Sperner' Lemma: For any subdivision of the simplex, any coloring respecting the boundary conditions must contain a *rainbow simplex.*



Question: Does Sperner's Lemma itself give some hardness of approximation result?

Rainbow-avoiding List Coloring

Problem: *Given k-uniform hypergraph $H = (V, E)$, color lists $L(v) \subseteq [k]$, find a coloring $\ell(v) \in L(v)$ that minimizes the number of rainbow hyperedges (all k colors).*

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We need one more ingredient:

- a simplicial subdivision with a "fractional coloring": on each cell we have a local distribution over colorings $\sum_\alpha y_{e,\alpha} = 1$;
- we design a fractional coloring using α with only 2 colors for a cell.
- on the other hand, every feasible coloring has a rainbow cell.

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Hardness result: [Mirzakhani-V. '14] *Assuming the UGC, there is no constant factor approximation for the Rainbow-avoiding List Coloring problem.*

Open questions

1. Find a generic rounding scheme for the Local Distribution LP.
2. Multiway Cut: improve the lower bound? (denser subdivisions of the simplex, using some Sperner coloring ideas?)
3. Gaps for other related problems: 0-extension, Hypergraph Multiway Partition, Submodular Labeling...
4. “Sperner’s Lemma for Intermediate Number of Colors": *Given a Sperner coloring of a simplicial subdivision in dimension k, what is the minimum possible number of cells containing at least d colors?*