The Limit Checker Number of a Graph

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The Checker Number is an invariant of a graph defined as the result of a game played on its vertices. Studying its behavior on the powers of a graph, we found that it's surprisingly similar to the well-known Shannon capacity and we defined an analogous quantity—the Limit Checker Number. Finally, we proved that this number can be calculated as a simple weight function of the original graph.

The Game

The Checker Number of a graph was introduced by Joseph Zaks in [1]. For a connected graph G, the checker number $\mathrm{CN}(G)$ is defined as the result of the following game:

(1) We start with one pebble at each vertex of the graph
(2) The allowed move is: we take a pebble at vertex u and jump over its neighbor v; the neighboring pebble v is removed and our pebble arrives at another vertex w which is adjacent to v. There must be at least one pebble at both u and v; negative pebble counts are not allowed, but there can be more than one pebble at one vertex.
(3) The goal of the game is to accumulate as many pebbles as possible at a given vertex; the checker number is then such a number of pebbles that can be accumulated at any vertex of the graph. We denote this number $\mathrm{CN}(G)$. Explicitely denote by $\mathrm{CN}(G,x)$ the largest number of pebbles that can be gathered at vertex x. Then $\mathrm{CN}(G) = \min_{x\in G} \mathrm{CN}(G,x)$.

Basic Facts

To get an upper bound for $\mathrm{CN}(G)$, we define a "weight" of the graph:

$$\mathrm{W}(G,x) = \sum_{v\in G} \sigma^{d(x,v)}$$
$$\mathrm{W}(G) = \min_{x\in G} \mathrm{W}(G,x),$$

where $d(x,v)$ is the distance between x and v in G and σ is the positive solution of the quadratic equation $\sigma^2+\sigma-1=0$ ($\sigma \doteq 0.618$).

We can also extend this definition to reflect the current state of the game by counting $\sigma^{d(x,v)}$ not over vertices, but over all pebbles with their respective distances from the goal vertex x. Then the weight of the final state is at least the number of pebbles in the desired position (distance $=0$), while the weight of the initial state is exactly $\mathrm{W}(G,x)$. But due to the careful choice of σ, the weight can only decrease or stay constant during the game. So we get an inequality $\mathrm{CN}(G,x) \le \mathrm{W}(G,x)$ and therefore

$$\mathrm{CN}(G) \le \mathrm{W}(G).$$

However, the weight of the graph doesn't provide any lower bound for $\mathrm{CN}(G)$; for example if G is a star with n vertices, then $\mathrm{W}(G) = 1+\sigma+(n-2)\sigma^2$, but $\mathrm{CN}(G)=1$. So for arbitrarily large $\mathrm{W}(G)$, $\mathrm{CN}(G)$ can still be constant.

We study the checker number of graphs generated as cartesian products of G; we shall see that the structure of these graphs makes it possible to exploit the graph weight much more efficiently.

The following facts have been proven by Joseph Zaks.

Claim.
(1) $CN(G \square H) \geq CN(G)CN(H)$
(2) $W(G \square H) = W(G)W(H)$

Proof: (1) Given a vertex (x, y) in $G \square H$, we can simply apply the process leading to $\mathrm{CN}(G)$ pebbles at x in each copy of G, and then play the game in the x-copy of H, $\mathrm{CN}(G)$-times; each time getting $\mathrm{CN}(H)$ pebbles at (x, y).
(2)

$$\begin{aligned}\mathrm{W}(G \square H, (x, y)) &= \sum_{v \in G, w \in H} \sigma^{d((x,y),(v,w))} = \sum_{v,w} \sigma^{d(x,v)+d(y,w)} \\ &= \sum_{v} \sigma^{d(x,v)} \sum_{w} \sigma^{d(y,w)} = \mathrm{W}(G, x)\mathrm{W}(H, y)\end{aligned}$$

□

The Limit Checker Number

Now consider the n-th power of G (G^0 is defined as a single vertex, and $G^{n+1} = G^n \square G$). We know that

$$\mathrm{CN}(G)^n \leq \mathrm{CN}(G^n) \leq \mathrm{W}(G^n) = \mathrm{W}(G)^n,$$

i.e. the checker number grows at least as the n-th power of $\mathrm{CN}(G)$, but at most as the n-th power of $\mathrm{W}(G)$; so it seems reasonable to study the behaviour of the n-th root of $\mathrm{CN}(G^n)$, which we shall denote by $\mathrm{C}(n)$. Also the following definition resembles the Shannon capacity of a graph (see [2]).

Definition. *Our object of interest will be the limit of $C(n)$ (if it exists), which we call the limit checker number:*

$$LCN(G) = \lim_{n\to\infty} C(n), \qquad \text{where} \qquad C(n) = \sqrt[n]{CN(G^n)}$$

By rewriting the equation above we get:

$$\mathrm{CN}(G) \leq \mathrm{C}(n) \leq \mathrm{W}(G)$$

The following is the main result of this paper; for its proof we will need two Lemmas.

Theorem. *For any connected graph G, $\quad LCN(G) = W(G)$.*

Lemma 1. *$LCN(G)$ always exists and $LCN(G) = \sup C(n)$.*

Proof: For any $m \geq kn$:

$$\mathrm{CN}(G^m) \geq \mathrm{CN}(G^{kn}) \geq \mathrm{CN}(G^n)^k.$$

If $kn \leq m < (k+1)n$,

$$\mathrm{C}(m) = \sqrt[m]{\mathrm{CN}(G^m)} \geq \sqrt[(k+1)n]{\mathrm{CN}(G^n)^k} = \mathrm{CN}(G^n)^{\frac{k}{n(k+1)}} = \mathrm{C}(n)^{k/(k+1)}.$$

As $\lim_{k\to\infty} \mathrm{C}(n)^{k/(k+1)} = \mathrm{C}(n)$, for any $\varepsilon > 0$, there is a sufficiently large k, so that:

$$\begin{array}{ll}\text{for all } m \geq kn: & \mathrm{C}(m) \geq \mathrm{C}(n) - \varepsilon, \\ \text{therefore} & \liminf \mathrm{C}(m) \geq \mathrm{C}(n)\end{array}$$

This holds for any n, thus we get:

$$\liminf \mathrm{C}(m) \geq \sup \mathrm{C}(n) \geq \limsup \mathrm{C}(n)$$

which implies $\lim \mathrm{C}(m) = \sup \mathrm{C}(n)$. □

Corollary. *CN*$(G) \leq$ *LCN*$(G) \leq$ *W*(G)

Lemma 2. *For any G connected where CN$(G) \geq 2$, there is a constant $\alpha > 0$, so that for any $n \geq 1$:*

$$CN(G^{n+\lfloor \alpha n \rfloor}) \geq \frac{\sigma}{2} CN(G^n) W(G^{\lfloor \alpha n \rfloor})$$

Proof: As $\mathrm{CN}(G) \geq 2$, we can choose $\alpha > 0$ so small that

$$\mathrm{CN}(G)\sigma^{\alpha D} > \tau,$$

where D is the diameter of G (the largest distance of two vertices), and $\tau = 1/\sigma \doteq 1.618 < 2$ is the golden ratio. Note that for $\alpha n < 1$, the statement of the lemma is trivial, so we can suppose $\alpha n \geq 1$.

In order to accumulate the desired number of pebbles, we follow the following strategy: We regard $G^{n+\lfloor \alpha n \rfloor}$ as the cartesian product of $H = G^n$, and $K = G^{\lfloor \alpha n \rfloor}$. Suppose our goal vertex is $[x; y]$, $x \in V(H), y \in V(K)$.

First, in each copy of H, denoted by $H_v, v \in V(K)$, we are able to move $\mathrm{CN}(H)$ pebbles to the vertex $[x; v]$. However, instead of this, by omitting some of the final moves arriving at $[x; v]$, we can distribute the $\mathrm{CN}(H)$ pebbles arbitrarily between $[x; v]$ and its neighbors in H_v. We can do this in such a manner that we get at least $\lfloor \sigma \mathrm{CN}(H) \rfloor$ pebbles at $[x; v]$, and $\lfloor \sigma^2 \mathrm{CN}(H) \rfloor$ pebbles in the neighborhood of $[x; v]$ (because $\sigma + \sigma^2 = 1$).

Our next step is to transport as many of these pebbles as possible to the final vertex $[x; y]$. To do this, we will follow the shortest path from $[x; v]$ to $[x; y]$ in the appropriate copy of K, and jump with the two groups of pebbles alternately over each other. (For the first move, we can assume that we have $\lfloor \sigma^2 \mathrm{CN}(H) \rfloor$ pebbles at a single neighbor of $[x; v]$ instead of being distributed over different neighbors.)

Denote the distance between vertices v and y in K by $d_K(v, y)$. By induction, we prove that after j steps, we get at least $\lfloor \sigma^{j+1} \mathrm{CN}(H) \rfloor$ pebbles at distance $d_K(v, y) - j$ from $[x; y]$, and at least $\lfloor \sigma^{j+2} \mathrm{CN}(H) \rfloor$ pebbles at distance $d_K(v, y) - j + 1$.

At the beginning (j=0), we have $\lfloor \sigma \mathrm{CN}(H) \rfloor$ pebbles at distance $d_K(v, y)$ and $\lfloor \sigma^2 \mathrm{CN}(H) \rfloor$ pebbles at distance $d_K(v, y) + 1$.

Assuming the induction hypothesis is true for $j \geq 0$, we jump with the $\lfloor \sigma^{j+2} \mathrm{CN}(H) \rfloor$ pebbles to the next vertex along the shortest path towards $[x; y]$, which is at distance $d_K(v, y) - j - 1$. Out of the $\lfloor \sigma^{j+1} \mathrm{CN}(H) \rfloor$ pebbles at distance $d_K(v, y) - j$, there will remain

$$\lfloor \sigma^{j+1} \mathrm{CN}(H) \rfloor - \lfloor \sigma^{j+2} \mathrm{CN}(H) \rfloor \geq \lfloor (\sigma^{j+1} - \sigma^{j+2}) \mathrm{CN}(H) \rfloor = \lfloor \sigma^{j+3} \mathrm{CN}(H) \rfloor$$

which is exactly what we need after $j + 1$ steps.

After $d_K(v, y)$ steps, we get at least

$$\lfloor \sigma^{d_K(v,y)+1} \mathrm{CN}(H) \rfloor \geq \frac{\sigma^{d_K(v,y)+1}}{2} \mathrm{CN}(H)$$

pebbles at $[x; y]$. The rounding operation cannot decrease the number of pebbles more than by half, because

$$\sigma^{d_K(x,v)+1} \mathrm{CN}(H) \geq \sigma^{\alpha n D+1} \mathrm{CN}(G)^n = \sigma(\sigma^{\alpha D} \mathrm{CN}(G))^n \geq \sigma \tau^n \geq 1$$

(to see this you should realize, that for any positive integer m and graf F the diameter of F^m equals m times the diameter of F).

In total, we get

$$\mathrm{CN}(H \,\square\, K) \geq \frac{\sigma}{2} \mathrm{CN}(H) \sum_{v \in V(K)} \sigma^{d_K(y,v)} \geq \frac{\sigma}{2} \mathrm{CN}(H) \mathrm{W}(K). \qquad \square$$

Now at last, we turn to the proof of the theorem.

Theorem. *For any G connected, $LCN(G) = W(G)$.*

Proof: If G is a single vertex, the equation is trivial.

If G contains at least two vertices (and is connected), then any vertex of G has a neighbor, any vertex of G^2 is contained in a C_4, and therefore $CN(G^2) \geq 2$. If we prove $LCN(G^2) = W(G^2)$, then

$$\mathrm{LCN}(G) = \sup_n \sqrt[n]{\mathrm{CN}(G^n)} \geq \sup_n \sqrt[2n]{\mathrm{CN}(G^{2n})} = \sqrt{\mathrm{LCN}(G^2)} = \sqrt{\mathrm{W}(G^2)} = \mathrm{W}(G)$$

so we can assume $\mathrm{CN}(G) \geq 2$.

For contradiction, let's suppose $\mathrm{LCN}(G) < \mathrm{W}(G)$, i.e.

$$\mathrm{LCN}(G) = \omega \mathrm{W}(G), 0 < \omega < 1$$

Then we'll find n such that:

$$\mathrm{C}(n) > \mathrm{LCN}(G) \sqrt[n]{\frac{2}{\sigma}\omega^{\lfloor \alpha n \rfloor}},$$

where $\mathrm{C}(n) = \sqrt[n]{\mathrm{CN}(G^n)}$, and α is the constant required in Lemma 2. This is possible, because:

$$\lim \sqrt[n]{\frac{2}{\sigma}\omega^{\lfloor \alpha n \rfloor}} = \omega^{\alpha} < 1,$$

so $$\mathrm{LCN}(G) \sqrt[n]{\frac{2}{\sigma}\omega^{\lfloor \alpha n \rfloor}} < \mathrm{LCN}(G) - \varepsilon,$$

whereas $$\mathrm{C}(n) > \mathrm{LCN}(G) - \varepsilon$$ for n sufficiently large.

So we have:

$$\begin{aligned} \mathrm{CN}(G^n) = \mathrm{C}(n)^n &> \mathrm{LCN}(G)^n \frac{2}{\sigma}\omega^{\lfloor \alpha n \rfloor} \\ &= (\omega \mathrm{W}(G))^n \frac{2}{\sigma}\omega^{\lfloor \alpha n \rfloor} \\ &= \frac{2}{\sigma}\mathrm{W}(G^n)\omega^{n+\lfloor \alpha n \rfloor} \end{aligned}$$

Using Lemma 2 for this n yields:

$$\mathrm{CN}(G^{n+\lfloor \alpha n \rfloor}) \geq \frac{\sigma}{2}\mathrm{CN}(G^n)\mathrm{W}(G^{\lfloor \alpha n \rfloor}) > \mathrm{W}(G^n)\omega^{n+\lfloor \alpha n \rfloor}\mathrm{W}(G^{\lfloor \alpha n \rfloor}) = (\mathrm{W}(G)\omega)^{n+\lfloor \alpha n \rfloor} = (\mathrm{LCN}(G))^{n+\lfloor \alpha n \rfloor}$$

i.e. $\mathrm{C}(n + \lfloor \alpha n \rfloor) > \mathrm{LCN}(G)$, which, however, contradicts the fact that $\mathrm{LCN}(G) = \sup \mathrm{C}(n)$. □

References:

[1] Joseph Zaks: The Checker Number of a Graph, 1996 (manuscript)

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