PTAS for Matroid Matching

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Why matroid matching?

Classical combinatorial optimization problems:

- Max-weight bipartite matching [Hungarian method, 1950's]
- Max-weight independent set in a matroid [Rado, 1950's]
- Max-weight non-bipartite matching [Edmonds, 1960's]
- Max-weight independent set in the *intersection of two matroids* [Edmonds/Lawler 1970's]

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Matroid matching:

- proposed by Lawler as a common generalization

Matroids

Definition

A matroid on $\mathcal{M} = (N, \mathcal{I})$ is a system of *independent sets* such that

1. $\emptyset \in \mathcal{I}$
2. $\forall J \in \mathcal{I}; I \subset J \Rightarrow I \in \mathcal{I}$
3. $\forall I, J \in \mathcal{I}; |I| < |J| \Rightarrow \exists j \in J \setminus I; I \cup \{j\} \in \mathcal{I}$.

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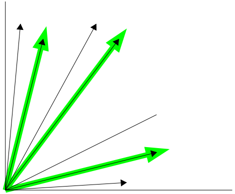
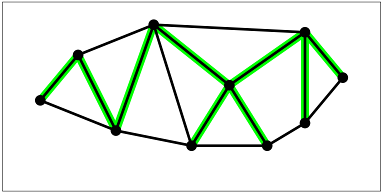
Examples:



linear matroid

(independent sets = linearly independent vectors)

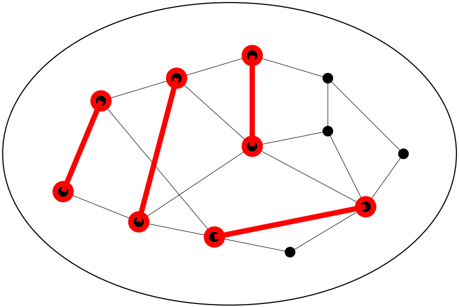


graphic matroid

(independent sets = acyclic subgraphs)

Matroid Matching

Given: Graph $G = (V, E)$, matroid $\mathcal{M} = (V, \mathcal{I})$.
Find: A matching M in G such that $V(M)$ is independent in $\mathcal{M}$.



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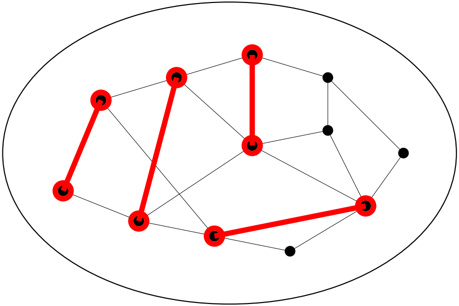
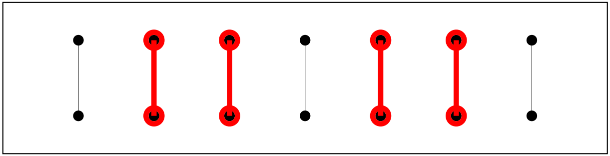


Note: Matroid matching is equivalent to its special case, where G itself is a matching.

Matroid parity

Given: Matroid $\mathcal{M} = (N, \mathcal{I})$, N partitioned into disjoint pairs $p_1, \ldots, p_n$.
Find: A subset $I \subseteq [n]$ such that $\bigcup_{i \in I} p_i$ is independent in $\mathcal{M}$.



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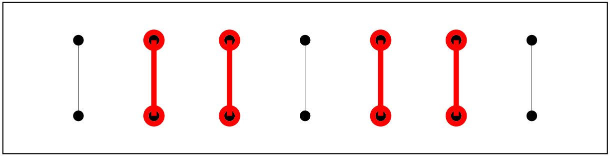
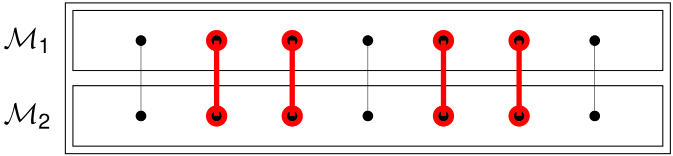


Reduction from matroid matching:

- Given $G = (V, E)$, replace each edge $e = (u, v)$ by two unique elements (u_e, v_e).
- For each vertex v, simulate the matching condition by defining $\{v_e : v \in e\}$ to be parallel copies of v in the matroid $\mathcal{M}$.

Special cases of matroid parity

Matroid intersection:
Given $\mathcal{M}_1, \mathcal{M}_2$, find the largest set independent in both matroids.



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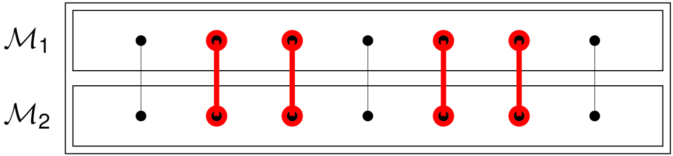
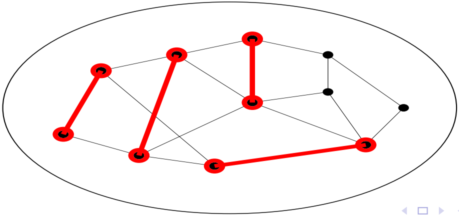
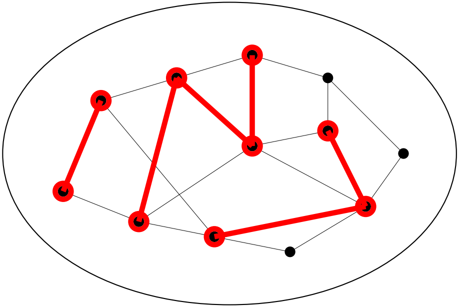


Matching in non-bipartite graphs:
obviously a special case of matroid matching.



The matchoid problem

Given: Graph $G = (V, E)$, matroid $\mathcal{M}_v = (E_v, \mathcal{I}_v)$ for each $v \in V$.
Find: A set of edges $F \subseteq E$ such that for each vertex $v \in V$, the incident edges $F \cap E_v$ are independent in $\mathcal{M}_v$.



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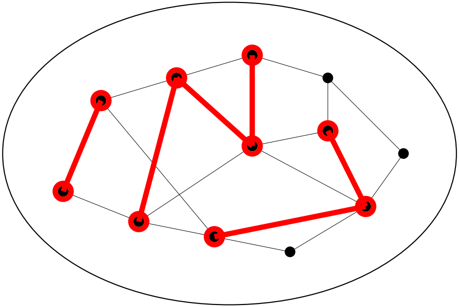
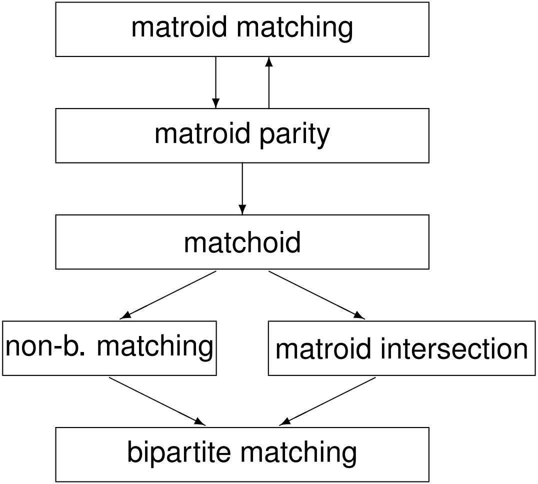


Note: The matchoid problem is a special case of matroid matching, and it still generalizes matroid intersection and non-bipartite matching.

Complexity status overview



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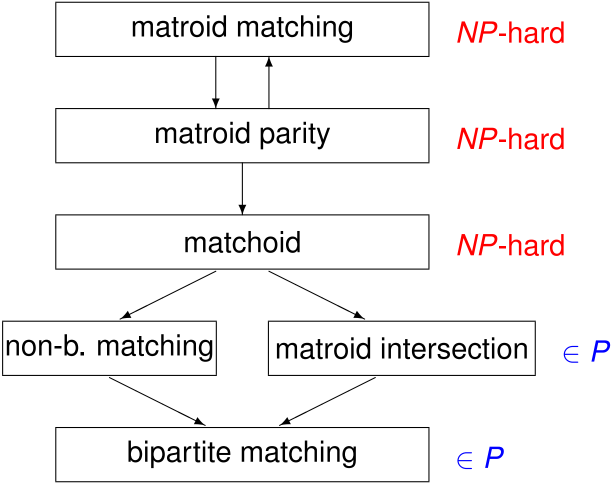

Good news

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A matroid $\mathcal{M} = (N, \mathcal{I})$ is linear if there are vectors $\{\mathbf{v}_i : i \in N)$ such that $I \in \mathcal{I}$ iff $\{\mathbf{v}_i : i \in I\}$ are linearly independent.

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Notes:

- There is also a randomized FPTAS in the weighted case for linear matroids.
- For general matroids given by an oracle, even unweighted matroid matching requires exponentially many queries to solve optimally.

Approximation?

Easy: The feasible solutions to a matroid matching problem form a 2-*independence system*: If A is feasible and $\{e\}$ is a feasible edge, then there are edges $a, b \in A$ such that $(A \setminus \{a, b\}) \cup \{e\}$ is feasible.

Jenkyns (1976): For any 2-independence system, the greedy algorithm gives a $1/2$-approximation (even in the weighted case).

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More generally, *matroid matching in k-uniform hypergraphs* is a k-independence system $\Longrightarrow$ $1/k$-approximation.

What we did

Theorem (Lee, Sviridenko, V.)

1. *There is a PTAS for unweighted matroid matching.*
2. *In k-uniform hypergraphs, $(2/k - \epsilon)$-approximation for any $\epsilon > 0$.*

Note: Special cases of k-uniform matroid matching are k-set packing ($2/k - \epsilon$ known by Hurkens-Schrijver) and intersection of k matroids (only $1/k$ known until recently).

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Open question: the weighted case.

Theorem (Lee, Sviridenko, V.)

A natural LP formulation of matroid matching has $\Omega(n)$ integrality gap. After r rounds of Sherali-Adams, still $\Omega(n/r)$ integrality gap.

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Lemma

If A, B are feasible solutions of matroid parity and

$$|A| < \left(1 - \frac{1}{2t}\right)|B|$$

then there is a local improvement for A with $s \leq 5^{t-1}$.

Base case: $t = 1$

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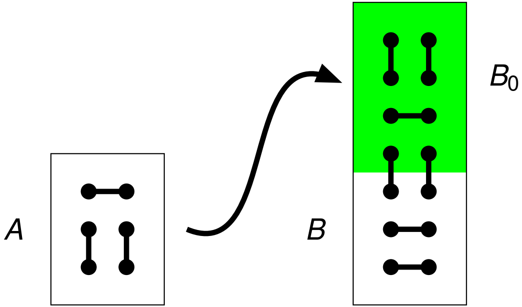
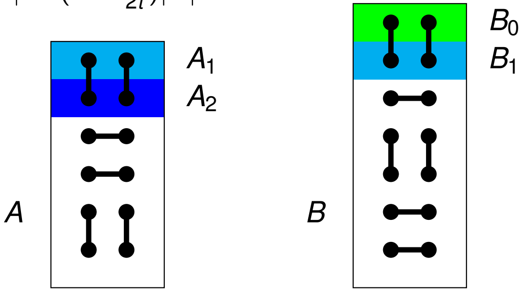


Proof:

- Since $|A| < \frac{1}{2}|B|$, we can extend A to an independent set $A \cup B_0$ such that $B_0 \subset B$ and $|B_0| > \frac{1}{2}|B|$.
- $|B_0| > |B \setminus B_0|$, so there must be a whole pair in B_0, which can be added to A.

General case: $t > 1$

We assume $|A| < (1 - \frac{1}{2t})|B|$.



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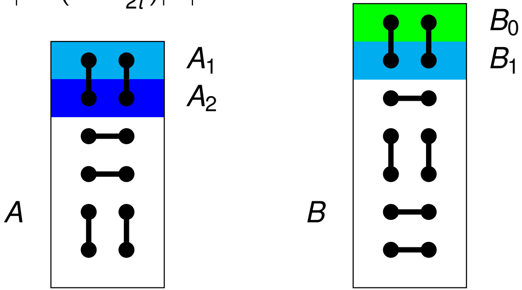

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- Let B_1 be paired up with B_0, and find $A_1 \subset A$ such that A_1 and B_1 can be swapped in the contracted matroid $\mathcal{M}/B_0$.

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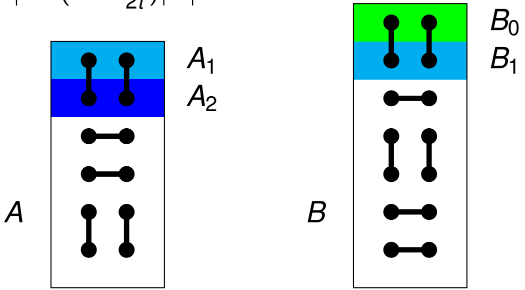

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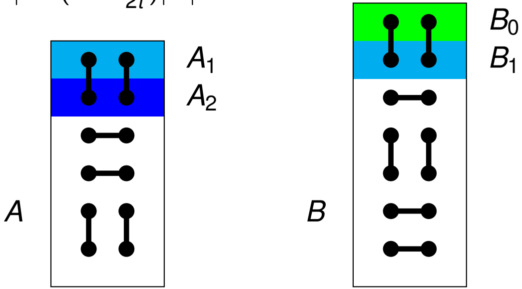

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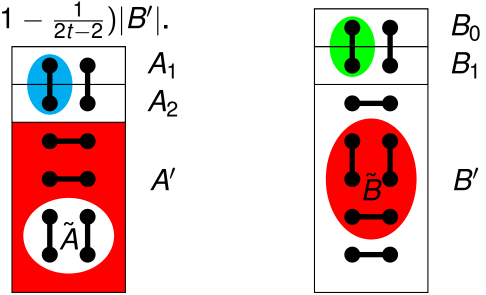
- Let B_1 be paired up with B_0, and find $A_1 \subset A$ such that A_1 and B_1 can be swapped in the contracted matroid $\mathcal{M}/B_0$.
- If A_1 contains a pair, kick it out and add two pairs from $B_0 \cup B_1$.
- Otherwise, let A_2 be paired up with A_1, and recurse on $A' = A \setminus (A_1 \cup A_2)$, $B' = B \setminus (B_0 \cup B_1)$ in $\mathcal{M}/(B_0 \cup B_1)$.

Inductive argument

We have $|A'| < (1 - \frac{1}{2t-2})|B'|$.



- By induction there is a local improvement $A' \setminus \tilde{A} \cup \tilde{B}$ in $\mathcal{M}/(B_0 \cup B_1)$ such that $|\tilde{B}| \le 5^{t-1}$.

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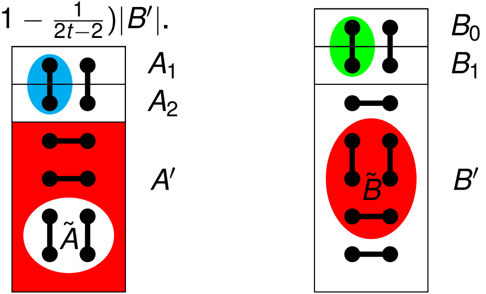

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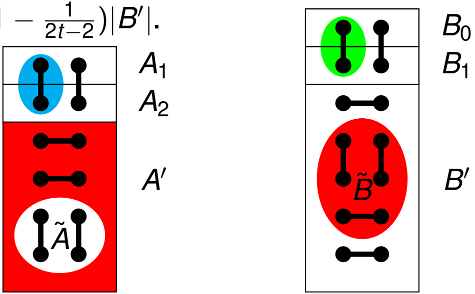

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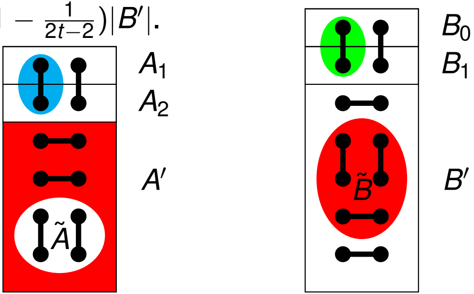

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- However, whatever we miss in $A_1 \cup A_2$, we can make up in $B_0 \cup B_1$.
- In total, we gain at least one pair and the swap size is $\leq 5|\tilde{B}| \leq 5^t$.

LP relaxations

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Natural LP:

$$\max \sum_{e \in E} w_e y_e :$$

$$\sum_{u \in S} x_u \le r_{\mathcal{M}}(S); \qquad \forall S \subseteq V$$

$$x_u = y_e; \quad \forall u \in e \in E,$$

$$x_u, y_e \ge 0.$$

Integrality gap

Example: Pairs $(u_1, v_1), \ldots, (u_n, v_n)$.
Matroid $\mathcal{M}_t$: a set S is independent if it contains at most t pairs, and any number of singletons from the remaining pairs.

$$\max \sum_{i=1}^{n} y_i :$$

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Optimal solution: $S = \{(u_1, v_1), (u_2, v_2), \ldots, (u_t, v_t)\}$, value t.
Fractional solution: $y_e = 1/2$ for each pair $e = (u_i, v_i)$, value $n/2$.

Sherali-Adams

Can we strengthen this LP?

Sherali-Adams hierarchy: For some $r \geq 1$, consider all disjoint $I, J \subseteq V$ such that $|I| + |J| \leq r$. Multiply each constraint by $\prod_{i \in I} y_i \prod_{j \in J} (1 - y_j)$, expand the products and replace each monomial $\prod_{\ell \in L} y_\ell^{k_\ell}$ by y_L.

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We get constraints like:

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$\rightarrow$ expand and replace monomials by y_L.

Fractional solution: $y_L = 1/2^{|L|}$ is still feasible for this LP, for $t = r$. $LP_r \geq n/2$, while $OPT = t = r$. Hence, the gap is $\Omega(n/r)$.

Conclusion

We showed a PTAS for *unweighted* matroid matching.
Is the following true?

For any $\epsilon > 0$*, there exists* $s(\epsilon)$ *such that local search with swap size* $s(\epsilon)$ *gives a* $(1 - \epsilon)$*-approximation for weighted matroid matching.*

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- More generally, can we get $(2/k - \epsilon)$-approximation for weighted k-uniform matroid matching?
- Or at least for the weighted matchoid problem?
  (we have a $2/3$-approximation for $k = 2$)