MATROID MATCHING: THE POWER OF LOCAL SEARCH

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Abstract. We consider the classical *matroid matching* problem. Unweighted matroid matching for linearly-represented matroids was solved by Lovász, and the problem is known to be intractable for general matroids. We present a PTAS for unweighted matroid matching for general matroids. In contrast, we show that natural LP relaxations that have been studied have an $\Omega(n)$ integrality gap and, moreover, $\Omega(n)$ rounds of the Sherali-Adams hierarchy are necessary to bring the gap down to a constant. More generally, for any fixed $k \geq 2$ and $\epsilon > 0$, we obtain a $(k/2+\epsilon)$-approximation for matroid matching in k-uniform hypergraphs, also known as the matroid k-parity problem. As a consequence, we obtain a $(k/2+\epsilon)$-approximation for the problem of finding the maximum-cardinality set in the intersection of k matroids. We also give a 3/2-approximation for the weighted version of a special case of matroid matching, the *matchoid problem*.

Key words. matroid, matching, local search, Sherali-Adams hierarchy, Chvátal closure.

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1. Introduction. The matroid matching problem is, given a graph $G = (V, E)$ and a matroid $\mathcal{M}$ on the vertex set V, find a matching M in G such that the vertices covered by M are independent in $\mathcal{M}$. This problem was proposed by Lawler as a common generalization of two important polynomial-time solvable problems: the non-bipartite matching problem, and the matroid-intersection problem (see [33]). Unfortunately, it turns out that matroid matching for general matroids (see [36, 27]) and even strongly base orderable matroids [50] is intractable and requires an exponential number of queries if the matroid is given by an oracle. This result can be easily transformed into an NP-completeness proof for a concrete class of matroids (see [48]). An important result of Lovász is that (unweighted) matroid matching can be solved in polynomial time for *linearly-represented matroids* (see [35]). There have been several attempts to generalize Lovász' result to the weighted case. Polynomial-time algorithms are known for some special cases of weighted matroid matching (see [53]), but for general linearly-represented matroids, there is only a pseudopolynomial-time randomized exact algorithm (see [8, 40]).

In this paper, we revisit the matroid matching problem for general matroids. Our main result is that a simple local-search algorithm gives a PTAS (in the unweighted case). This is the first PTAS for general matroid matching. On the other hand, we show that LP-based approaches including the Sherali-Adams hierarchy fail to provide any meaningful approximation. To the best of our knowledge, this is the first example of a problem where there is such a dramatic gap between the performance of the Sherali-Adams hierarchy and a simple combinatorial algorithm. Our algorithm requires only a membership oracle for the matroid $\mathcal{M}$. We also provide approximation results for a generalization of the problem to hypergraphs; more details follow.

Previously known algorithms for matroid matching apply only to linearly-represented matroids. That is, they require that a linear representation is available. This is a stronger property than just being linearly representable. For example, if we have linear representations of matroids $\mathcal{M}_1$ and $\mathcal{M}_2$ over a field $\mathbb{F}$, then it is known that for the *matroid union* $\mathcal{M}_1 \vee \mathcal{M}_2$, there exists a linear representation over a finite extension

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of $\mathbb{F}$. However, the typical way of defining such a representation is via extending $\mathbb{F}$ by indeterminates (see [43]), and it is not clear that this can be carried out efficiently. On the other hand, using the *matroid partitioning algorithm*, it is easy to efficiently realize an independence oracle for $\mathcal{M}_1 \vee \mathcal{M}_2$, so our approximation algorithms apply immediately and efficiently.

We assume familiarity with approximation algorithms (see [55], for example) and matroid algorithmics (see [48], for example). Throughout, we consider maximization problems. A *c-approximation algorithm* finds in polynomial time a solution of value at least OPT$/c$. Briefly, for a matroid $\mathcal{M}$, we denote the ground set of $\mathcal{M}$ by $V = V(\mathcal{M})$, its set of independent sets by $\mathcal{I} = \mathcal{I}(\mathcal{M})$, and its rank function by $r_{\mathcal{M}}$. For a given matroid $\mathcal{M}$, the associated *matroid constraint* on a set S is the condition $S \in \mathcal{I}(\mathcal{M})$ or equivalently $|S| = r_{\mathcal{M}}(S)$.

In the *matroid hypergraph matching problem*, we are given a matroid $\mathcal{M} = (V, \mathcal{I})$ and a hypergraph $G = (V, \mathcal{E})$ where $\mathcal{E} \subseteq 2^V$. Note that the vertex set of the hypergraph G and the ground set of the matroid $\mathcal{M}$ are the same. The goal is to choose a maximum-cardinality collection of disjoint hyperedges $E^* \subseteq \mathcal{E}$ in hypergraph G, such that the union of the hyperedges in E^* is an independent set in matroid $\mathcal{M}$. If G is a graph, we obtain the classical matroid matching problem.

The matroid hypergraph matching problem generalizes several classical optimization problems, namely:

1. *Hypergraph matching*, also known as *set packing*: Given a hypergraph G, find as many pairwise disjoint hyperedges as possible. This is easily seen to be a special case of (hypergraph) matroid matching, by taking $\mathcal{M}$ to be a *free matroid* (i.e., $\mathcal{I}(\mathcal{M}) = 2^V$), so the matroid independence condition is void. Hypergraph matching in general is NP-hard, but when G is a graph (every hyperedge has exactly two vertices), it is the classical matching problem which led Edmonds to the very notion of polynomial-time algorithms (see [16, 17]).
2. In the *k-matroid intersection problem* we are given k matroids $\mathcal{M}_1 = (V, \mathcal{I}_1), \ldots,$ $\mathcal{M}_k = (V, \mathcal{I}_k)$ on the same ground set V, and the goal is to find a maximum cardinality set S of elements that is independent in each of the k matroids, i.e., $S \in \cap_{j=1}^k \mathcal{I}_j$. The k-matroid intersection problem is NP-hard for $k \geq 3$ but polynomially solvable for $k = 2$ (see [48]).
3. A problem of intermediate generality is the *k-uniform matchoid* problem, defined for $k = 2$ by Edmonds and studied by Jenkyns (see [28]). In this problem, we have a k-uniform hypergraph and a matroid $\mathcal{M}_v$ given for each vertex v, having ground set the set of hyperedges containing v. The goal is to choose a maximum collection of hyperedges S, such that for each v, the hyperedges in S containing v form an independent set in $\mathcal{M}_v$. This can be also seen as a packing problem with many matroid constraints, where each item participates in at most k of them.
   By taking each $\mathcal{M}_v$ to be the uniform matroid of rank 1, we get the set-packing problem. By taking a (multi-)hypergraph on $V = \{1, 2, \ldots, k\}$ with n parallel hyperedges $e_1 = \ldots = e_n = \{1, 2, \ldots, k\}$, and a matroid $\mathcal{M}_i$ on ground set $[n]$ for each $i = 1, 2, \ldots, k$, we get k-matroid intersection. On the other hand, the matchoid problem is a special case of matroid matching, as we show below. We remark that even for $k = 2$, the matchoid problem is NP-hard (see [36]).
4. The special case of the matroid hypergraph matching problem when each vertex (i.e., element of the ground set) belongs to a unique hyperedge, and

all hyperedges have cardinality exactly k, is known as the *matroid k-parity problem*, or simply the *matroid parity problem* when $k = 2$. As we show below, this problem is in fact equivalent to k-uniform matroid matching, even in terms of approximation.

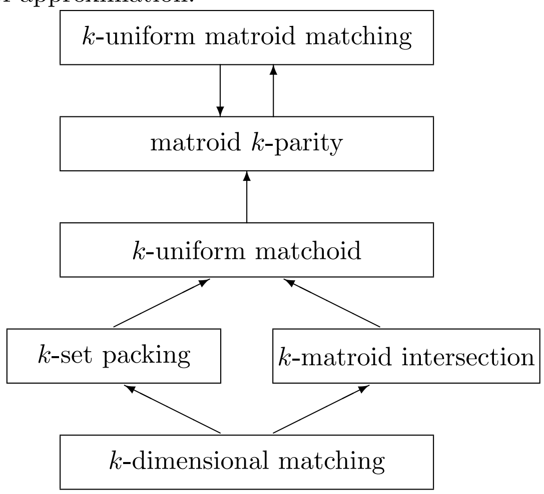

FIG. 1.1. *The hierarchy of k-uniform packing problems, with approximation-preserving reductions.*

Next, we explain how the k-uniform matchoid problem is a special case of matroid k-parity. Given a hypergraph G, we can replace each vertex by n_v distinct copies, where n_v is the number of hyperedges containing v. We replace each hyperedge in G by a collection of distinct copies of its elements, so that we get a hypergraph G' where the hyperedges are disjoint. In the matchoid problem, we have a matroid $\mathcal{M}_v$ defined on the n_v copies of each vertex v, and we define a new matroid $\mathcal{M}'$ by taking the union of the matroids $\mathcal{M}_v$. Then matroid k-parity for $(G', \mathcal{M}')$ is equivalent to the original k-uniform matchoid for $(G, \mathcal{M}_v)$.

A similar construction implies that matroid k-parity includes (and therefore is equivalent to) matroid matching in k-uniform hypergraphs: Given an instance of k-uniform matroid matching, we define n_v copies for each vertex v where n_v is the degree of v in the hypergraph $G = (V, \mathcal{E})$. We replace each hyperedge in G by a collection of distinct copies of its elements, so that the new hyperedges are disjoint. Let $G' = (V', \mathcal{E}')$ be the new hypergraph. We also define a new matroid $\mathcal{M}'$ on the ground set V', where the n_v copies of each vertex act as parallel copies. That is, a set of vertices $S' \subseteq V'$ is independent in $\mathcal{M}'$ if it contains at most one copy of each vertex from V and the respective set $S \subseteq V$ is independent in $\mathcal{M}$. It is not difficult to show that $\mathcal{M}'$ is a matroid.

Henceforth, we will discuss the matroid k-parity problem with the understanding that all our results can be easily extended to the k-uniform matroid matching problem. For the purposes of this paper, the terms *matroid matching* and *matroid parity* are essentially interchangeable.

Literature overview. There are a few different lines of research relevant to our results. The matroid parity problem (for $k = 2$) was originally popularized by Lawler [32]. The maximum-cardinality matroid parity problem was shown to have exponential query complexity in general (see [36, 27]), and to be NP-hard for some concrete classes of matroids. If the matroid is linearly represented, the problem is polynomially solvable (see [35, 21, 41, 42, 52, 13]). It is not known whether the weighted case is polynomially solvable for linear matroids, but randomized pseudopolynomial algorithms are known [8, 40]. The linearly-represented matroid k-parity problem also admits a polynomial-time algorithm if k is a constant, and the rank of matroid $\mathcal{M}$ is $O(\log |V|)$ (see [4]). The special case of matroid intersection can be solved in polynomial time for arbitrary matroids (see for example [48]), even in the weighted case. For more on matroid parity, applications and closely related problems, see [6, 37, 45, 46].

It is not difficult to show that a greedy algorithm that adds hyperedges one-by-one, as long as the current solution remains independent, gives a k-approximation for general weighted matroid k-parity and that this guarantee is tight. This follows from the work of Jenkyns on k-independence systems; see [29] (see also Section 2). Until our present work, this was the only algorithm known for the general matroid k-parity problem with a provable approximation guarantee. The only improvement to our knowledge has been achieved in the case of unweighted matroid matching ($k = 2$), where Fujito proved that a local-search algorithm gives a $3/2$-approximation (see [20]). Also, it should be noted that randomized pseudo-polynomial time exact algorithms for weighted matroid matching ($k = 2$) on linearly representable matroids [8, 40] can be transformed into randomized fully polynomial-time approximation schemes, and this has been used in work on the Steiner tree problem [44]. To summarize the situation for $k = 2$, weighted matroid matching for linearly representable matroids admits an FPRAS; for general matroids only a greedy 2-approximation is known.

Linear-programming relaxations for the matroid matching problem have been studied in [54, 11, 12, 22]. An LP proposed by Vande Vate (see [54]) has been shown to be half-integral, and moreover, there are polynomial-time algorithms to find a half-integral optimal solution in the unweighted case (see [11, 12]) and the general weighted case (see [22]). However, this approach has not yielded any approximation algorithms for matroid matching.

We now survey known results on the approximability of various special cases of the matroid k-parity problem. Local-search algorithms exploring a larger neighborhood for the k-set packing problem were analyzed in [26]. They showed that a local-search algorithm produces a solution with approximation guarantee $k/2 + \epsilon$ for any fixed $\epsilon > 0$ and constant k in polynomial time. Hardness of approximation results are known for the maximum k-dimensional matching problem which is a special case of the maximum k-set packing problem and hence also matroid k-parity. The best known lower bound is the $\Omega(k/\log k)$ hardness of approximation of [25]. It is also known that a large-neighborhood local-search algorithm has a tight approximation guarantee of $k - 1 + \epsilon$ for the problem of (weighted) k-matroid intersection (see [34]). It should be noted though that unweighted versions of packing problems seem to be easier for algorithm design and analysis, and that approximation guarantees for general linear objective functions can often be improved for unweighted variants.

The Sherali-Adams hierarchy (see [49]) has been studied recently for a number of combinatorial optimization problems (see [30, 31] for surveys on the results in this area). Mathieu and Sinclair proved (see [38]) for the non-bipartite matching problem that r rounds of Sherali-Adams applied to the matching polytope have integrality gap $1 + O(1/r)$ and hence provides a PTAS (while the problem can be solved exactly in polynomial time). Chan and Lau considered k-uniform hypergraph matching, i.e., k-set packing, and proved that that even after $\Theta(n)$ rounds of Sherali-Adams, the standard LP has integrality gap at least $k - 2$ (see [9]). In contrast, local search techniques yield a $(k/2 + \epsilon)$-approximation (see [26]), and an alternative LP (using intuition from local search) has integrality gap at most $(k + 1)/2$ (see [9]).

Our Results. We prove that a very simple local-search algorithm gives a PTAS for unweighted matroid parity. Given the negative results for the matroid parity problem (see [36, 27]), this is the best type of worst-case result we could expect for this problem.

On the negative side, we show that the known linear relaxations of matroid parity do not yield any reasonable approximation guarantee (even for $k = 2$ with unit weights). More precisely, all variants of the LPs that have been proposed have an $\Omega(n)$ integrality gap for instances with n pairs. Moreover, $\Omega(n)$ rounds of the Sherali-Adams hierarchy are required to generate an LP with a constant integrality gap. Additionally, we demonstrate that the Chvátal rank of the polytope is $\Omega(n)$. This is a strong manifestation of the fact that LP-based hierarchies do not always match the performance of combinatorial algorithms (which was, in a weaker sense, shown previously for the matching problem in graphs (see [38]) and hypergraphs (see [9])).

For the more general problem of unweighted matroid k-parity, we present a $(k/2+\epsilon)$-approximation, for any fixed $k \geq 2$ and $\epsilon > 0$. As a special case, this subsumes the (unweighted) k-matroid intersection problem for which a k-approximation was known since 1976 (see [29]) and has been only recently improved to $k - 1 + \epsilon$ (see [34]).

The algorithm that we analyze is simple local search that in each iteration seeks to remove $s(\epsilon)$ hyperedges and add $s(\epsilon)+1$ hyperedges to the current solution in such a way that the new solution defines an independent set in matroid $\mathcal{M}$. We call this the *s-neighborhood local-search algorithm.* If there is no improvement the algorithm stops and outputs the current local optimum. Our analysis uses an idea from [26] to inductively reduce the instance, and given the performance of the local search on the smaller instance, to derive the guarantee on the original one. But the presence of the matroid independence constraint complicates matters significantly. In particular, to achieve the approximation guarantee of $1+\epsilon$ for the matroid parity problem we need to implement the local-search algorithm with $s(\epsilon)$ exponentially large in $1/\epsilon$, while it is well known that $s(\epsilon) = \lceil 1 + 1/\epsilon \rceil$ is enough for the maximum matching problem in graphs (the fixed-size augmenting-path algorithm could be viewed as a local-search algorithm). We do not know at this point if having such a large neighborhood is necessary or if it is just an artifact of our analysis. Surprisingly, for $k \geq 3$, we show that to achieve $k/2 + \epsilon$ approximation, it is enough to run the local-search algorithm with $s(\epsilon)$ polynomially bounded in $1/\epsilon$.

We have also developed a 3/2-approximation algorithm for the weighted matchoid problem, which is a special case of weighted matroid parity. This result uses an LP relaxation of the matchoid problem and its known half-integrality (see [11, 12, 22, 54] for closely related LPs). We provide an alternative proof that is very simple and intuitive and which might be of independent interest.

The rest of the paper is organized as follows. In §2, we show that matroid k-parity is a special case of a "k-independence system" which implies a greedy k-approximation. In §3, we present our PTAS for unweighted matroid parity. In §4, we consider various linear-programming relaxations for the matroid parity problem and present our lower bounds on their integrality gap. In §5, we present a $(k/2+\epsilon)$-approximation for matroid k-parity. In §6, we present our 3/2-approximation algorithm for the weighted matchoid problem.

2. Relation to independence systems. First, we show that the matroid k-parity falls in the framework of k-independence systems (see [29]). Such systems generalize intersections of k matroids, and in fact several definitions of various degrees of generality have been proposed (see also [39, 7]). Jenkyns' definition is as follows.

Definition 2.1. *For a family of sets $\mathcal{I} \subset 2^V$ and a set $W \subseteq V$, we define a* base *of W to be any inclusion-wise maximal subset $B \subseteq W$ such that $B \in \mathcal{I}$. We call $\mathcal{I}$ a* p-system, *if for any $W \subseteq V$,*

$$\max_{B:\text{base of } W} |B| \leq p \cdot \min_{B:\text{base of } W} |B|.$$

Lemma 2.2. *The independence system corresponding to matroid k-parity is a k-system.*

Proof. Consider an independent collection of hyperedges $W = \{e_1, \dots, e_\ell\}$, and two bases (i.e., setwise maximal independent subsets) B_1, B_2 of W. Assume toward a contradiction that $|B_2| > k|B_1|$. Let $S_1 = \bigcup\{e : e \in B_1\}$ and $S_2 = \bigcup\{e : e \in B_2\}$; i.e., $|S_i| = k|B_i|$ and both S_1 and S_2 are independent in the matroid $\mathcal{M}$. By the matroid extension axiom, S_1 can be extended by elements of S_2 to a set $S_1 \cup S_2'$ independent in $\mathcal{M}$, where $S_2' \subseteq S_2 \setminus S_1$ and $|S_2'| = |S_2| - |S_1| = k|B_2| - k|B_1|$. Note that S_2' is not necessarily a union of hyperedges. However, it must contain at least one hyperedge, otherwise $|S_2'| \leq (k-1)|B_2| < k|B_2| - k|B_1|$. Therefore, there is a hyperedge $e_i \in B_2 \setminus B_1$ that we can add to B_1 which contradicts the fact that B_1 is a base of W. □

The work of [28, 19] for p-systems gives the following results (see also [7]).

Theorem 2.3. *The greedy algorithm gives a p-approximation for maximizing a linear function over a p-system. Moreover, the greedy algorithm gives a $(p+1)$-approximation for maximizing a monotone submodular function over a p-system.*

Corollary 2.4. *The greedy algorithm gives a k-approximation for matroid k-parity even in the weighted version. Moreover, the greedy algorithm gives a $(k+1)$-approximation for maximizing a monotone submodular function over sets feasible for the matroid k-parity problem.*

We regard the greedy k-approximation for weighted matroid k-parity as a "folklore" result and a starting point for further improvements. For unweighted matroid parity ($k = 2$), this has been improved to a factor of 3/2 by Fujito [20]. For general k, no better approximation was known prior to our work.

3. PTAS for matroid parity. Let us start with the case of $k = 2$, i.e., matroid parity. In an instance of matroid parity, we have disjoint pairs, and we look for a maximum-cardinality collection of pairs whose union forms an independent set in a given matroid. We present a PTAS for this problem.

Definition 3.1. *For feasible solutions A and B of matroid parity, a "local move of size s between A and B" is a choice of $s-1$ pairs $e_1, \dots, e_{s-1}$ inside A, and s pairs $e_1', \dots, e_s'$ inside B, such that $(A \setminus \bigcup_{i=1}^{s-1} e_i) \cup \bigcup_{i=1}^{s} e_i'$ is again feasible.*

Theorem 3.2. *For any $\epsilon > 0$, a local-search algorithm that considers local moves of size up to $s(\epsilon) = 5^{\lceil 1/(2\epsilon) \rceil}$ achieves a $(1+\epsilon)$-approximation for the matroid parity problem.*

The same result also holds for matroid matching, by the simple reduction that we outlined in the introduction. The theorem follows immediately from the following characterization of local optima.

Lemma 3.3. *Let $t \geq 1$, and A, B feasible solutions to the matroid parity problem such that*

$$|A| < \left(1 - \frac{1}{2t}\right)|B|.$$

Then there is a local move of size at most 5^{t-1} between A and B.

Assuming that B is an actual optimum and A is a local optimum with respect to local moves of size 5^{t-1}, this implies that A is a $2t/(2t-1)$-approximate solution. This means that for any fixed $\epsilon > 0$, we can pick $t = \lceil 1/(2\epsilon) \rceil + 1$ and $s = 5^{t-1}$; the corresponding local-search algorithm achieves a $(1+\epsilon)$-approximation for matroid parity.

It remains to prove the lemma. Our proof uses the standard notion of matroid contraction. For a set $S \subset V(\mathcal{M})$, $\mathcal{M}/S$ (read $\mathcal{M}$ *contract* S) is the matroid having ground set $V(\mathcal{M}) \setminus S$ and set of independent sets $\{T \subseteq V(\mathcal{M}) \setminus S \;:\; T \cup J \in \mathcal{I}(\mathcal{M})\}$, where J is an arbitrary maximal independent subset of S with respect to $\mathcal{M}$.

Proof. Let A, B be feasible solutions as above. (We assume for simplicity that A and B are disjoint, otherwise we can contract the intersection, which only decreases the ratio $|A|/|B|$.) Because $|A| < |B|$, there exists $B_0 \subset B, |B_0| = |B| - |A|$, such that $A \cup B_0$ is independent in $\mathcal{M}$. We proceed by induction on t.

Base case. $t = 1$. For $t = 1$, we have $|A| < \frac{1}{2}|B|$. Then, $|B_0| = |B| - |A| > \frac{1}{2}|B|$. Because B decomposes into disjoint pairs, this means there must be a pair contained inside B_0. This pair can be added to A without violating independence, i.e., there is a local move of size one.

General case. $t \geq 2$. We assume that $|A| = |B| - a$ where $a > \frac{1}{2t}|B|$. We also assume $a \leq \frac{1}{2}|B|$, as otherwise we are in the base case. We construct a set $B_0 \subset B$ as above, with $A \cup B_0$ independent and $|B_0| = a$. Again, if there is a pair contained inside B_0, we can add it to A, and we are done. So let us assume that no pair is contained completely inside B_0.

Since B is a union of disjoint pairs, this means that every pair intersecting B_0 also contains an element in $B \setminus B_0$, Let us denote the elements matched with B_0 by B_1. We have $|B_1| = |B_0| = a$. Let $\mathcal{M}_0 = \mathcal{M}/B_0$ denote the matroid where B_0 has been contracted. Because $A \cup B_0$ and $B_1 \cup B_0$ are independent in $\mathcal{M}$ (by construction), we get that A and B_1 are independent in $\mathcal{M}_0$. Because $|A| = |B| - a \geq a = |B_1|$, we can extend B_1 by adding (possibly zero) elements from A, to form an $\mathcal{M}_0$-independent set of cardinality $|A|$: i.e., there is $A_1 \subseteq A$ such that $(A \setminus A_1) \cup B_1$ is independent in $\mathcal{M}_0$ and $|A_1| = |B_1| = a$.

If A_1 contains a pair e then we can find a local move as follows: $A \setminus e$ is independent in $\mathcal{M}_0$, and so is the set $(A \setminus A_1) \cup B_1$, and $|(A \setminus A_1) \cup B_1| \geq |A \setminus e| + 2$. Therefore, $A \setminus e$ can be extended to a set $(A \setminus e) \cup \{x', x''\}$ independent in $\mathcal{M}_0$, such that $x', x'' \in B_1$. The elements x', x'' are contained in pairs e', e'' whose remaining elements are in B_0. Because $(A \setminus e) \cup \{x', x''\}$ is independent in $\mathcal{M}_0 = \mathcal{M}/B_0$, any elements of B_0 can be added for free, and $(A \setminus e) \cup e' \cup e''$ is independent in $\mathcal{M}$. This defines a local move of size two.

The rest of the proof deals with the case in which there is no pair contained in A_1. Then, every pair intersecting A_1 also contains an element in $A \setminus A_1$; let us denote the elements matched with A_1 by A_2. We have $|A_2| = |A_1| = a$. Let us define $A^* = A \setminus (A_1 \cup A_2)$ and $B^* = B \setminus (B_0 \cup B_1)$.

Our goal now is to apply the inductive hypothesis to A^* and B^*, which are obtained from A and B by removing the same number of elements ($2a$). Therefore, the ratio between their cardinalities increases. By the inductive hypothesis, there is a "small" local move between A^* and B^*. Using matroid operations, we are going to extend this to a local move between A and B.

The inductive step. We define a new matroid $\mathcal{M}_1 = \mathcal{M}_0/B_1 = \mathcal{M}/(B_0 \cup B_1)$. By construction, the sets $A^* = A \setminus (A_1 \cup A_2)$ and $B^* = B \setminus (B_0 \cup B_1)$ are both independent in $\mathcal{M}_1$. They both form a union of pairs and hence are feasible solutions to the matroid parity problem for $\mathcal{M}_1$. We have $|A^*| = |A| - 2a$ and $|B^*| = |B| - 2a$. Because $|A| = |B| - a$, we get

$$\frac{|A^*|}{|B^*|} = \frac{|A| - 2a}{|B| - 2a} = \frac{|B| - 3a}{|B| - 2a} = 1 - \frac{1}{|B|/a - 2}.$$

Because we assumed $a > \frac{1}{2t}|B|$, we have $|B|/a < 2t$ and $|A^*| < (1 - \frac{1}{2t-2})|B^*|$, so we can apply the inductive hypothesis. There is a local move of size $s = 5^{t-2}$ between A^* and B^*, i.e., a union of $s-1$ pairs $\tilde{A} \subseteq A^*$ and s pairs $\tilde{B} \subseteq B^*$ such that $(A^* \setminus \tilde{A}) \cup \tilde{B}$ is independent in $\mathcal{M}_1$. Our goal is to find a local move of size $5s = 5^{t-1}$ between A and B (in $\mathcal{M}$).

The set $(A^* \setminus \tilde{A}) \cup \tilde{B}$ is independent in $\mathcal{M}_1$. Unfortunately, $(A \setminus \tilde{A}) \cup \tilde{B}$ is not necessarily independent, even in $\mathcal{M}$. We have to proceed more carefully. The set $(A^* \setminus \tilde{A}) \cup A_2 = A \setminus (A_1 \cup \tilde{A})$ is independent in $\mathcal{M}_1 = \mathcal{M}_0/B_1$, because $(A \setminus A_1) \cup B_1$ was constructed to be independent in $\mathcal{M}_0$. If $|A_2| > |\tilde{B}|$, we can extend $(A^* \setminus \tilde{A}) \cup \tilde{B}$ to a set $(A^* \setminus \tilde{A}) \cup \tilde{B} \cup (A_2 \setminus C_2)$ independent in $\mathcal{M}_1$, where $C_2 \subseteq A_2$ and $|C_2| \leq |\tilde{B}|$. If $|A_2| \leq |\tilde{B}|$, we set $C_2 = A_2$ and $(A^* \setminus \tilde{A}) \cup \tilde{B} \cup (A_2 \setminus C_2)$ is again independent in $\mathcal{M}_1$. In both cases, we have $|C_2| \leq |\tilde{B}|$.

The new set $(A^* \setminus \tilde{A}) \cup \tilde{B} \cup (A_2 \setminus C_2)$ is also independent in $\mathcal{M}_0$ (a weaker condition). So is $(A^* \setminus \tilde{A}) \cup (A_2 \setminus C_2) \cup A_1$, as any subset of A is independent in $\mathcal{M}_0$. Therefore, we can extend $(A^* \setminus \tilde{A}) \cup \tilde{B} \cup (A_2 \setminus C_2)$ to a set $(A^* \setminus \tilde{A}) \cup \tilde{B} \cup (A_2 \setminus C_2) \cup (A_1 \setminus C_1)$ in $\mathcal{M}_0$, where $C_1 \subseteq A_1$ and $|C_1| \leq |\tilde{B}|$.

The set we have obtained is not necessarily a union of pairs, so let us remove all the pairs containing some element of $C_1 \cup C_2$. Let us denote by C' the union of these pairs. By our construction, we have $C_1 \cup C_2 \subseteq C' \subseteq A_1 \cup A_2$. Further, let us define $C'_1 = C' \cap A_1$ and $C'_2 = C' \cap A_2$. Each pair in $A_1 \cup A_2$ contains exactly one element in A_1 and one element in A_2; therefore $|C'_1| = |C'_2|$. Also, $|C'_1| = |C'_2| \leq |C_1 \cup C_2|$, because each element of $C_1 \cup C_2$ contributes at most one pair to C'.

We obtain a feasible solution $A^+ = (A^* \setminus \tilde{A}) \cup \tilde{B} \cup (A_2 \setminus C'_2) \cup (A_1 \setminus C'_1)$ in $\mathcal{M}_0$ (see Figure 2). Now, consider the set $(A^+ \setminus A_1) \cup B_1 = (A^* \setminus \tilde{A}) \cup \tilde{B} \cup (A_2 \setminus C'_2) \cup B_1$. This is independent in $\mathcal{M}_0$, because $A^+ \setminus A_1 = (A^* \setminus \tilde{A}) \cup \tilde{B} \cup (A_2 \setminus C'_2)$ was constructed to be independent in $\mathcal{M}_1 = \mathcal{M}_0/B_1$. If $C'_1 \neq \emptyset$, A^+ misses some elements in A_1 and the cardinality of $(A^+ \setminus A_1) \cup B_1$ is larger than $|A^+|$, because $|(A^+ \setminus A_1) \cup B_1| = |A^+| - |A_1 \setminus C'_1| + |B_1| = |A^+| + |C'_1|$. Hence, we can extend A^+ by $|C'_1|$ elements of B_1, let us call them F_1, to obtain a set $A^+ \cup F_1$ independent in $\mathcal{M}_0$. (If $C'_1 = \emptyset$, we continue with $F_1 = \emptyset$.) Each pair touching F_1 has exactly 1 element in B_1 and the other element in B_0. Let F_0 be the elements of B_0 matched with F_1. We can add F_0 for free and obtain an independent set $A^+ \cup F_1 \cup F_0$ in $\mathcal{M}$. We have $|F_0| = |F_1| = |C'_1| = |C'_2|$. Now $A^+ \cup F_1 \cup F_0$ is a union of pairs and hence a feasible solution, of cardinality

$$|A^+ \cup F_1 \cup F_0| = |A^+ \cup C'_1 \cup C'_2| = |(A \setminus \tilde{A}) \cup \tilde{B}| > |A|.$$

Finally, let us estimate the size of this local move. We removed $\tilde{A} \cup C'_1 \cup C'_2$ from A, and added $\tilde{B} \cup F_1 \cup F_0$ instead. The size of C'_1 is bounded by $|C'_1| \leq |C_1 \cup C_2| \leq 2|\tilde{B}|$, hence $|F_1| = |C'_1| \leq 2|\tilde{B}|$. The size of F_0 is equal to the size of F_1, i.e., $|F_0 \cup F_1| = 2|F_1| \leq 4|\tilde{B}|$. In summary, we are adding at most $5|\tilde{B}|$ elements to A, i.e., the size of the local move is at most $5|\tilde{B}| = 5s = 5^{t-1}$. ◻

4. LP Relaxations. In this section, we consider a linear-programming approach to matroid parity. Our results in this direction are mostly negative and indicate that

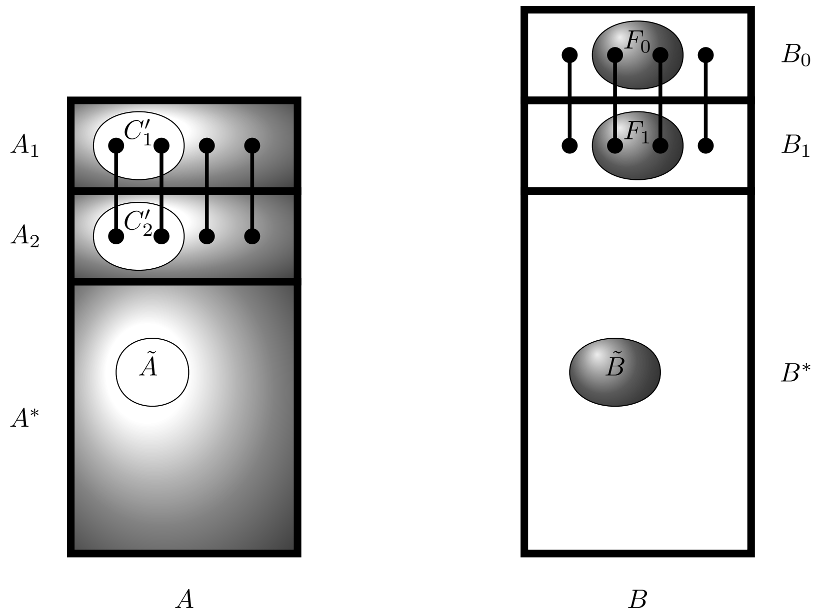

FIG. 3.1. *The inductive step: We assume that there is a local move between A^* and B^* (namely, replace $\tilde{A}$ by $\tilde{B}$), and we extend it to a local move between A and B.*

linear programming in this case fails very badly compared to the local-search algorithm presented in the previous section. We formulate our linear programs for the general case of matroid k-parity but the case $k = 2$ is already sufficiently general to obtain our results.

We start with the following natural LP for the weighted matroid k-parity problem (equivalent to an LP studied in [22]). The variables y_e correspond to the hyperedges and the variables x_u correspond to the elements of the ground set. We assume here that each hyperedge alone is independent; otherwise we remove it from the instance.

$$\max \sum_{e \in \mathcal{E}} w_e y_e, \tag{4.1}$$

$$\sum_{u \in S} x_u \le r_{\mathcal{M}}(S), \forall S \subseteq V, \tag{4.2}$$

$$x_u = y_e, \forall u \in e, e \in \mathcal{E}, \tag{4.3}$$

$$x_u, y_e \ge 0, \forall u \in V, e \in \mathcal{E}. \tag{4.4}$$

In the objective function (4.1) we are maximizing the total weight of chosen hyperedges. The constraints (4.3) correspond to the fact that we can choose a hyperedge only if we chose all its vertices. The constraints (4.2) are the standard rank constraints for the matroid $\mathcal{M}$ and any independent set of vertices must satisfy them.

For any set of elements $S \subseteq V$ let $\mathrm{sp}(S) = \{u \in V \mid r_{\mathcal{M}}(S \cup \{u\}) = r_{\mathcal{M}}(S)\}$ be the *span* of S in matroid $\mathcal{M}$. (Clearly, we always have $\mathrm{sp}(S) \supseteq S$.) $F \subseteq V$ is a *flat* if $\mathrm{sp}(F) = F$ (for more information on these concepts, see [48], Chapter 39). Because $r_{\mathcal{M}}(\mathrm{sp}(S)) = r_{\mathcal{M}}(S)$, it is easy to see that it is enough to write the constraints (4.2) for every flat $F \subseteq V$; the inequality for arbitrary $S \subset V$ is implied by the flat $F = \mathrm{sp}(S)$.

Another set of valid inequalities (for $k = 2$) were suggested by Vande Vate [54] and studied in subsequent work [11, 12, 22, 54]. For a set $S \subseteq V$ and a hyperedge e, let $a(S, e) = r_{\mathcal{M}}(S \cap \mathrm{sp}(e))$. In case of a flat F, the intuition is that $a(F, e)$ is the dimension of the subspace of F generated by e. The LP proposed by Vande Vate is as follows.

$$\max \sum_{e \in \mathcal{E}} w_e y_e, \tag{4.5}$$

$$\sum_{e \in \mathcal{E}} a(S, e) y_e \leq r_{\mathcal{M}}(S), \forall S \subseteq V, \tag{4.6}$$

$$y_e \geq 0, \forall e \in \mathcal{E}. \tag{4.7}$$

Again, it is equivalent to consider the inequalities (4.6) only for flats, which was the formulation given by Vande Vate. This LP is potentially stronger than LP (4.1-4.4), which can be equivalently obtained from (4.5-4.7) by replacing $a(S, e)$ with the quantity $|S \cap e|$ which is no larger.

It is known that the linear program (4.5-4.7) is half-integral in the 2-uniform case. Moreover, there are polynomial time algorithms to find a half-integral optimal solution in the unweighted 2-uniform case (see [11, 12]) and the weighted 2-uniform case (see [22]). The following lemma shows the validity of the LP (4.5–4.7) in the general k-uniform case. We could not find a published proof of validity (even for $k = 2$), so for completeness we provide a short proof.

Lemma 4.1. *The inequalities (4.6) are valid for the matroid k-parity problem.*

Proof. Consider any feasible solution, a collection of hyperedges $E^* = \{e_1, \ldots, e_k\}$ such that $e_1 \cup \ldots \cup e_k$ is an independent set in matroid $\mathcal{M}$. In the following, we denote the rank function of $\mathcal{M}$ simply by $r(S)$. Let $S_i = S \cap \mathrm{sp}(e_i)$. Note that $r(S_i) = a(S, e_i)$. We claim that for any $i \leq k$, $r(S_1 \cup \ldots \cup S_i) = r(S_1 \cup \ldots \cup S_{i-1}) + r(S_i)$. By induction, we will get that $r(S_1 \cup \ldots \cup S_k) = \sum_{i=1}^{k} r(S_i) = \sum_{i=1}^{k} a(S, e_i)$ which implies the inequalities (4.6).

We let $r_A(S) = r(A \cup S) - r(A)$; due to submodularity, this is a non-increasing function of A. Let $A = S_1 \cup \ldots \cup S_{i-1} = S \cap (\mathrm{sp}(e_1) \cup \ldots \cup \mathrm{sp}(e_{i-1}))$ and $B = e_1 \cup \ldots \cup e_{i-1}$. Our goal is to prove that $r_A(S_i) = r(S_i)$. Because $A \subseteq \mathrm{sp}(B)$, we get $r_A(S_i) \geq r_{\mathrm{sp}(B)}(S_i) = r_B(S_i)$, using that $r(\mathrm{sp}(B)) = r(B)$ and $r(S_i \cup \mathrm{sp}(B)) = r(S_i \cup B)$, by the definition of span and submodularity of r. On the other hand, as $B \cup e_i$ is an independent set, and $S_i \subseteq \mathrm{sp}(e_i)$, we also have

$$\begin{aligned} r(\mathrm{sp}(e_i)) = r_B(\mathrm{sp}(e_i)) &= r_B(S_i) + r_{B \cup S_i}(\mathrm{sp}(e_i))) \\ &\leq r_A(S_i) + r_{S_i}(\mathrm{sp}(e_i)) \end{aligned}$$

using again the submodularity of r. This implies that $r_A(S_i) \geq r(\mathrm{sp}(e_i)) - r_{S_i}(\mathrm{sp}(e_i)) = r(S_i)$. The opposite inequality is obvious and hence $r_A(S_i) = r(S_i)$. □

In the following, we use examples where $e = \mathrm{sp}(e)$ for all hyperedges $e \in \mathcal{E}$. Note that in this case, $a(S, e) = r_{\mathcal{M}}(S \cap \mathrm{sp}(e)) = r_{\mathcal{M}}(S \cap e) = |S \cap e|$, and hence the two LPs are in fact equivalent.

4.1. Integrality gap example. It is known that the integrality gap of the LP relaxation (4.1–4.4) is $k - 1 + \frac{1}{k}$ for the maximum-weight k-uniform hypergraph matching problem [9]. Therefore, it is tempting to conjecture that a similar result should hold for matroid hypergraph matching. Unfortunately, as we show below, the integrality gap of the linear-programming relaxation (4.1–4.4) is $\Omega(|\mathcal{E}|)$ even when $k = 2$ and the matroid is linear over the rationals.

Example. Consider a ground set $V = \{u_1, v_1, \dots, u_n, v_n\}$ of size $2n$, partitioned into pairs $e_i = \{u_i, v_i\}$. Given an integer parameter $t \geq 0$, we define a matroid $\mathcal{M}_t = (V, \mathcal{I})$ as follows. For a set $S \subseteq V$, let $p(S)$ be the number of pairs e_i such that $e_i \subseteq S$. Then let $S \in \mathcal{I}$ if $p(S) \leq t$. It can be checked that $\mathcal{I}$ satisfies the matroid independence axioms: For any $S, T \in \mathcal{I}$, $|S| < |T|$, either T contains an element from a pair $\{u_i, v_i\}$ which is disjoint from S, or it contains more pairs than S. In either case, we can extend S by adding some element of T.

For $t = 0$, the matroid $\mathcal{M}_0$ is a simple partition matroid — a set is independent if no more than one element is selected from each pair. For $t > 0$, the matroid $\mathcal{M}_t$ is related to $\mathcal{M}_0$ via the following (not very well known) operation. Generally, for a matroid $\mathcal{M}$ of rank r on ground set V and an integer p satisfying $r \leq p \leq |V|$, the *elongation of* $\mathcal{M}$ *to height* p has as its family of bases the sets that have rank r in $\mathcal{M}$ and have cardinality p. Note that elongation is dual to the more familiar matroid operation of truncation (see [56, pp. 59–60]). The matroid $\mathcal{M}_t$ is the elongation of $\mathcal{M}_0$ to height $n + t$. Although there is a standard way of passing from a matrix representation of a matroid $\mathcal{M}$ to a matrix representation of an elongation of $\mathcal{M}$, the general procedure does not preserve the field that $\mathcal{M}$ is represented over (see [57, p. 402]). Instead, for $\mathcal{M}_t$, we give a direct representation over the rationals.

PROPOSITION 4.2. *The matroid $\mathcal{M}_t$ is linear over the rationals. Moreover the representation has bit size that is polynomial in n.*

Proof. Take W to be a $t \times n$ Vandermonde matrix. That is, we take distinct positive integers $c_1, c_2, \dots, c_n$ (we can take $c_j = j$). Then, we define column j of W to be $(c_j^0, c_j^1, c_j^2, ..., c_j^{t-1})$. Our representation of the matroid $\mathcal{M}_t$ is

$$\left(\begin{array}{c|c} I & I \\ \hline W & 2W \end{array}\right)$$

where the first block of columns represents $u_1, \dots, u_n$ and the second block represents $v_1, \dots, v_n$.

Next, we consider any $S \subseteq V$ and we check that the respective columns of this purported representation are linearly independent if and only if S is independent in $\mathcal{M}_t$.

If S contains $t + 1$ pairs, we take the respective columns and delete all-zero rows. We obtain a $(t + 1 + t) \times 2(t + 1) = (2t + 1) \times (2t + 2)$ submatrix, which clearly has dependent columns as it has more columns than rows.

Suppose that S contains at most t pairs - we can assume it is a base that contains exactly t pairs and $n - t$ singletons. First, for each of the $n - t$ singletons, the corresponding column has a non-zero in some row and all other columns have a zero in that row. So we can delete such columns and the associated rows, and concentrate on the columns corresponding to just the pairs. So we are led to a $2t \times 2t$ submatrix of the form

$$\left(\begin{array}{c|c} I & I \\ \hline A & 2A \end{array}\right),$$

where A is a $t \times t$ submatrix of W. We want to argue now that this $2t \times 2t$ matrix is non-singular. After elementary row operations, we pass to the matrix

$$\left(\begin{array}{c|c} I & I \\ \hline 0 & A \end{array}\right).$$

Now it is clear that we just need to check that A is non-singular. But the matrix A is itself a Vandermonde matrix, which is well known to be non-singular.

Finally, taking $c_j = j$, $j = 1, 2, \ldots, n$, the largest integer used is just $2n^{t-1}$, which has polynomial bit size. ◻

Next, we consider the LP for this family of examples. We have variables y_i for $i = 1, \ldots, n$, which are constrained by $y_i \in [0, 1]$. Because $\mathrm{sp}(\{e_i\}) = \{e_i\}$, the LPs (4.1-4.4) and (4.5-4.7) coincide. It is enough to write the constraints (4.6) for flats, and in particular only for sets of the form $S = \bigcup_{i \in T} e_i$ for some T. This is because including only one element of a pair in S always increases $r_{\mathcal{M}}(S)$ by 1 and hence cannot strengthen the constraint. Also, for $S = \bigcup_{i \in T} e_i$ where $|T| \le t$, the rank is $r_{\mathcal{M}}(S) = |S|$ and the respective constraint (4.6) is implied by $y_i \le 1$. The only non-trivial constraints are for $S = \bigcup_{i \in T} e_i$, $|T| > t$, where we get $r_{\mathcal{M}}(S) = 2t + (|T| - t) = t + |T|$. Also, $a(S, e_i) = 2$ for all $i \in T$. Therefore, the LP is as follows.

$$\max \sum_{i=1}^{n} w_i y_i, \tag{4.8}$$

$$\sum_{i \in T} y_i \le \frac{1}{2}(t + |T|), \forall T \subseteq [n], |T| > t \tag{4.9}$$

$$0 \le y_i \le 1, \forall i. \tag{4.10}$$

Lemma 4.3. *The integrality gap of LP (4.8–4.10) is at least $n/(2t)$, even in the unweighted case.*

Proof. It is easy to see that $y_i = 1/2$ for all $i = 1, \ldots, n$ is a feasible fractional solution. Therefore, $LP \ge n/2$. However, only t pairs can be selected in an integral optimum, i.e., $OPT = t$. ◻

For $t = 1$, we get an $\Omega(n)$ integrality gap. One way to improve the quality of linear-programming relaxations is to add valid inequalities that cut bad fractional solutions. One of the possible classes of valid inequalities are the so-called *clique inequalities* that were recently shown to reduce the integrality gap for unweighted hypergraph matching from $k - 1$ to $(k + 1)/2$ [9]. This motivates us to define the undirected graph $G' = (\mathcal{E}, E')$ where the vertices are the hyperedges $e \in \mathcal{E}$ in our instance of matroid hypergraph matching and the edges are defined between "incompatible hyperedges" e and e', i.e., when $r(e \cup e') < |e \cup e'|$. A set of vertices C in graph G' is called a clique if it has an edge between every pair of vertices in C. Let $\mathcal{C}$ be the set of all cliques in graph G'. Then the following set of constraints is valid for the matroid hypergraph matching problem

$$\sum_{e \in C} y_e \le 1, \quad \forall C \in \mathcal{C}. \tag{4.11}$$

However, as we can see in the example above (for $t \ge 2$, where any two hyperedges are compatible), sometimes G' is empty and the clique inequalities do not add any non-trivial constraints. More generally, we could add all the valid constraints for the stable-set polytope corresponding to G' (or perhaps consider the semidefinite program corresponding to the Lovász θ-function). The relaxation would still remain the same in our example.

In the next section, we consider the strongest known systematic way of generating valid constraints in linear programming, namely the *Sherali-Adams hierarchy.*

4.2. The Sherali-Adams hierarchy. The Sherali-Adams hierarchy produces progressively stronger refinements of a given LP by introducing new variables y_L indexed by subsets of the original variables, and then projecting back to the space of the original variables. We follow the formalism of [38]. To carry out r rounds of Sherali-Adams, we consider all pairs of disjoint subsets I, J of variables such that $|I \cup J| = r$. We multiply each constraint by $\prod_{i \in I} y_i \prod_{j \in J}(1 - y_j)$, expand all the monomial terms and replace every square y_i^2 by y_i. Now all terms are multilinear, and we replace each occurrence of $\prod_{\ell \in L} y_\ell$ by a new variable y_L. We also do the same for the constraint $\prod_{i \in I} y_i \prod_{j \in J}(1-y_j) \geq 0$, for all disjoint I, J such that $|I \cup J| = r+1$. This defines the new LP; note that the variables y_L for $|L| > 1$ play no role in the objective function and thus the polytope can be viewed as projected back to the original space.

LEMMA 4.4. *The integrality gap of LP (4.1-4.4) still remains at least $n/(2r)$ after r rounds of the Sherali-Adams hierarchy.*

Proof. Our starting point is LP (4.8–4.10), which is a special case of LP (4.1-4.4) for the example we discussed above. The parameter t is chosen equal to the desired number of rounds, $t = r$. We have constraints $\sum_{\ell \in T} y_\ell \leq \frac{1}{2}(r + |T|)$ for all $|T| > r$. We multiply this constraint by $\prod_{i \in I} y_i \prod_{j \in J}(1 - y_j)$ and obtain

$$\sum_{\ell \in T} y_\ell \prod_{i \in I} y_i \prod_{j \in J}(1 - y_j) \leq \frac{1}{2}(r + |T|) \prod_{i \in I} y_i \prod_{j \in J}(1 - y_j). \tag{4.12}$$

We expand the products, linearize the expressions, and replace monomials $\prod_{\ell \in L} y_\ell$ by new variables y_L as explained above. We also do the same thing for the constraints $\prod_{i \in I} y_i \prod_{j \in J}(1 - y_j) \geq 0$ with $|I \cup J| = r + 1$. We claim that $y_L = 1/2^{|L|}$ for all $|L| \leq r + 1$ is a feasible solution for the new LP.

To see this, first observe that whenever we have $\ell \in J$ on the left-hand side of (4.12), the corresponding term contains a factor of $y_\ell(1 - y_\ell) = y_\ell - y_\ell^2$ which disappears after linearization. In terms where $\ell \in I$, we get y_ℓ^2 which gets linearized to y_ℓ. Equivalently, we can replace y_ℓ by 1 in its appearance before the product $\prod_{i \in I} y_i$, whenever $\ell \in I$. Variables outside of $I \cup J$ remain unchanged. Therefore, after linearization, the left-hand side is equal to $(|T \cap I| + \sum_{\ell \in T \setminus (I \cup J)} y_\ell) \prod_{i \in I} y_i \prod_{j \in J}(1 - y_j)$.

Now we replace the monomials $\prod_{\ell \in L} y_\ell$ by y_L and substitute $y_L = 1/2^{|L|}$. Note that this is equivalent to directly substituting $y_\ell = 1/2$ for all ℓ. Thus the left-hand side becomes

$$\begin{aligned} &(|T \cap I| + \frac{1}{2}|T \setminus (I \cup J)|)\, 2^{-|I \cup J|} \\ = &\frac{1}{2}(|T \cap I| + |T \setminus J|)\, 2^{-|I \cup J|} \leq \frac{1}{2}(r + |T|)\, 2^{-|I \cup J|} \end{aligned}$$

using the fact that $|I| \leq r$. This verifies the linearized form of constraint (4.12).

The inequalities arising from $\prod_{i \in I} y_i \prod_{j \in J}(1 - y_j) \geq 0$ are easy to verify, because our assignment $y_L = 1/2^{|L|}$ is equivalent to substituting $y_i = 1/2$. Therefore, our fractional solution is feasible for r rounds of Sherali-Adams.

Finally, the value of our fractional solution is equal to $n/2$, because each singleton variable is $y_i = y_{\{i\}} = 1/2$. The integral optimum is $OPT = r$. ☐

To summarize, our LP (4.8–4.10) is an instance of the strongest "natural LP" for matroid matching we are aware of, namely the Vande Vate LP (4.5-4.7). The same LP

(4.8–4.10) is obtained even with the added clique constraints (4.11) and other valid constraints for the stable-set polytope which are hard to optimize over in general. On top of this LP, we apply the Sherali-Adams hierarchy and the gap still remains superconstant for $o(n)$ rounds.

4.3. Lower bound on Chvátal rank. Another popular way to derive progressively stronger linear-programming relaxations is to apply Chvátal-Gomory cuts (for example, see [47]). Let $P := \{x \in \mathbb{R}^n \ : \ \sum_{j=1}^n a_{i,j} x_j \le b_i, \text{ for } i = 1, \dots, m\}$ be a polyhedron defined by rational data. Let $\sum_j a_j x_j \le b$ be a linear inequality satisfied by all points in P. Such an inequality is called a *valid (linear) inequality* for P. Now, let $P_I := \mathrm{conv}\,(P \cap \mathbb{Z}^n)$. If $\sum_j a_j x_j \le b$ is a valid linear inequality for P, and $a_j \in \mathbb{Z}$ for all j, then clearly $\sum_j a_j x_j \le \lfloor b \rfloor$ is a valid linear inequality for P_I. Any such inequality is known as a *Chvátal-Gomory cut* with respect to P. Applying all Chvátal-Gomory cuts to P, we obtain the *first Chvátal closure* $P^{(1)}$ of P. It is a classical result that $P^{(1)}$ is again a polyhedron; that is, only a finite number of the Chvátal-Gomory cuts for P are needed to describe $P^{(1)}$. Now, we can repeatedly apply this closure operator, and we obtain after r repetitions the *r-th Chvátal closure* $P^{(r)}$ of P. If the linear inequality $\sum_j a_j x_j \le b$ is valid for $P^{(r)}$ but not $P^{(r-1)}$, then we say that the inequality has Chvátal rank r. It is a classical result that when P is described by rational data, $P_I = P^{(r)}$ for some finite r. Hartmann established lower bounds on Chvátal rank for many classes of polytopes of interest in combinatorial optimization (see [23] and also [15]). Eisenbrand and Schulz established that for polytopes in $[0,1]^n$, the Chvátal rank is at most $3n^2 \log(n)$, and furthermore that there is a family of polytopes in $[0,1]^n$ that has Chvátal rank at least $(1+\epsilon)n$ for some fixed $\epsilon > 0$ (see [18]). Singh and Talwar showed in [51] that Chvátal-Gomory cuts help reduce the integrality gap for hypergraph matching.

The proof of the following lemma is a generalization of the geometric proof from [15] of the classical result by Chvátal [14] that a clique inequality involving n variables requires at least $\lfloor \log_2 n \rfloor$ rounds of the Chvátal-Gomory hierarchy. Let $\mathcal{P}_t$ be the polytope described by (4.8–4.10). So $\mathcal{P}_t^{(r)}$ is its r-th Chvátal closure.

LEMMA 4.5. *The point $y^r \in \mathbb{R}^n$ defined by $y_i^r = \frac{1}{2}(\frac{t}{t+1})^r$, for $i = 1, \dots, n$, is in $\mathcal{P}_t^{(r)}$.*

Proof. We prove the lemma by induction on the number r of rounds of the Chvátal closure. The base case $r = 0$ is trivial because the solution $(1/2, \dots, 1/2)$ is obviously feasible for the linear program (4.8–4.10) for any parameter $t \ge 1$.

We assume that $y^{r-1} \in \mathcal{P}_t^{(r-1)}$. Let $\sum_{i=1}^n a_i y_i \le b$ be a valid linear inequality for $\mathcal{P}_t^{(r-1)}$, with $a_i \in \mathbb{Z}$. Because $(0, \dots, 0)$ is a feasible integral solution for the linear program (4.8–4.10), we obtain $(0, \dots, 0) \in \mathcal{P}_t^{(r-1)}$, and therefore $b \ge 0$. If $a_i \le 0$ for all $i = 1, \dots, n$, then $\sum_{i=1}^n a_i y_i^r \le 0 \le \lfloor b \rfloor \le b$.

Otherwise, let $A^+ := \sum_{i | a_i > 0} a_i$; we assume $A^+ > 0$. Note that $\sum_{i=1}^n a_i y_i^r \le \sum_{i|a_i>0} a_i y_i^r < A^+$, because $0 < y_i^r < 1$. If the vector $(a_1, \dots, a_n)$ has at most t strictly positive components, then $A^+ \le b$ because any integral solution with at most t variables equal to one and the remaining variables equal to zero is a feasible integral solution for the linear program (4.8–4.10), which implies that such a solution must belong to $\mathcal{P}_t^{(r-1)}$. Moreover, because A^+ is an integer, we have $A^+ \le \lfloor b \rfloor$. Therefore, we have $\sum_{i=1}^n a_i y_i^r < A^+ \le \lfloor b \rfloor$.

Consider now an inequality in which the vector $(a_1, \dots, a_n)$ has more than t strictly positive components. Recall that $a_i \in \mathbb{Z}$ and hence these coefficients are at least 1. An integral solution χ having any t of the respective coordinates equal to one and remaining coordinates equal to zero is feasible for (4.8–4.10); therefore $t \le \sum_{i=1}^n a_i \chi_i \le b$. We obtain

$$\sum_{i=1}^{n} a_i y_i^r = \frac{t}{t+1} \sum_{i=1}^{n} a_i y_i^{r-1} \le \frac{t}{t+1} b \le \lfloor b \rfloor,$$

using $t \le b$ in the last inequality, which implies that $y^r \in \mathcal{P}^{(r)}$. ∎

COROLLARY 4.6. *The integrality gap of the linear-programming relaxation obtained from the linear-programming relaxation (4.8–4.10) after t rounds of Chvátal closure is $\Omega(n/t)$.*

Proof. By Lemma 4.5 there exists a feasible fractional solution of value $\frac{n}{2}(\frac{t}{t+1})^t = \Omega(n)$ in $\mathcal{P}_t^{(t)}$, while the value of the integral optimal solution is t. ∎

COROLLARY 4.7. *The Chvátal rank of the polytope defined by (4.8–4.10) with $t = \lfloor \frac{n}{2e} \rfloor$ is at least $\lfloor \frac{n}{2e} \rfloor$.*

Proof. Consider the inequality $\sum_{i=1}^n y_i \le t$. This inequality is valid for all integer feasible solutions of the linear-programming relaxation (4.8–4.10). Let r be the smallest integer such that this inequality is valid for $\mathcal{P}_t^{(r)}$; i.e., r is the Chvátal rank of the inequality $\sum_{i=1}^n y_i \le t$. By Lemma 4.5, we get $\frac{n}{2}(\frac{t}{t+1})^r \le t$. Therefore,

$$r \ge \frac{\ln(\frac{n}{2t})}{\ln(1+\frac{1}{t})} \ge t \ln\left(\frac{n}{2t}\right).$$

For $t = \lfloor \frac{n}{2e} \rfloor$, we have $\ln(\frac{n}{2t}) \ge 1$, and the Chvátal rank is $r \ge \lfloor \frac{n}{2e} \rfloor$. ∎

5. Matroid k-parity.

Here we extend the analysis of local search to matroid k-parity; i.e., instead of pairs, we work with hyperedges of size k. We assume that $k \ge 3$. In an instance of matroid k-parity, all hyperedges are mutually disjoint. We remark that our analysis extends to k-uniform matroid matching where hyperedges need not be disjoint, by a standard reduction as we discussed in Section 1.

Interestingly, the analysis for $k \ge 3$ is slightly different and the complexity of our $(k/2+\epsilon)$-approximation for $k \ge 3$ has a much better dependence on ϵ than our PTAS for $k = 2$ (matroid matching). More precisely, while we need local moves of size exponential in $1/\epsilon$ in order to achieve a $(1+\epsilon)$-approximation for matroid matching, local moves of size polynomial in $1/\epsilon$ are sufficient to achieve a $1/(2/k-\epsilon)$-approximation for matroid k-parity. We do not know whether our analysis is optimal in terms of this dependence.

DEFINITION 5.1. *For feasible solutions A, B of matroid k-parity, a* local move *of size s between A and B is a choice of $s-1$ hyperedges $e_1, \dots, e_{s-1}$ from A, and s hyperedges $e'_1, \dots, e'_s$ from B, such that $(A \setminus \bigcup_{i=1}^{s-1} e_i) \cup \bigcup_{i=1}^{s} e'_i$ is feasible.*

THEOREM 5.2. *For any $k \ge 3$ and $\epsilon > 0$, a local-search algorithm that considers local moves of size up to $s(\epsilon) = \lceil 1/\epsilon^3 \rceil$ achieves a $1/(2/k-\epsilon)$-approximation for the matroid k-parity problem.*

This follows easily from the following characterization of local optima.

LEMMA 5.3. *Let $k \ge 3$, $t \ge 1$, and A, B feasible solutions to the matroid k-parity problem such that*

$$|A| < \left(\frac{2}{k} - \frac{1}{(k-1)^t}\right)|B|.$$

Then there exists a local move of size at most $(2k+1)^{t-1}$ between A and B.

Note that in order to achieve a $1/(2/k-\epsilon)$-approximation, it suffices to pick $t = \lceil \log_{k-1}(1/\epsilon) \rceil$ and $s(\epsilon) = (2k+1)^{t-1} \leq 1/\epsilon^{\log_{k-1}(2k+1)}$. Then, if A is a local optimum and B is a global optimum, the lemma implies that $|A| \geq (2/k - 1/(k-1)^t)|B| \geq (2/k-\epsilon)|B|$. For simplicity, we replaced $1/\epsilon^{\log_{k-1}(2k+1)}$ by $1/\epsilon^3$ (considering $k \geq 3$) in the statement of the theorem, but for large k the dependency gets close to $1/\epsilon$.

It remains to prove the lemma.

Proof. (Lemma 5.3). Let A, B be feasible solutions as above. (We assume for simplicity that A and B are disjoint, otherwise we can contract the intersection, which only decreases the ratio $|A|/|B|$.) Because $|A| < |B|$, there exists $B' \subset B, |B'| = |B| - |A|$ such that $A \cup B'$ is independent in $\mathcal{M}$. We proceed by induction on t.

Base case: $t = 1$. Here, we have $|A| < (\frac{2}{k} - \frac{1}{k-1})|B| < \frac{1}{k}|B|$. Then, it is impossible that every hyperedge in B contains some element in $B \setminus B'$, because that would mean that $|A| = |B \setminus B'| \geq \frac{1}{k}|B|$. Hence, there must be a hyperedge contained completely inside B', which can be added to A without violating independence. This means there is a local move of size one.

General case: $t \geq 2$. We assume that $|A| = (2/k - \epsilon)|B|$ and $\epsilon > \frac{1}{(k-1)^t}$. Again, if there is a hyperedge contained inside B', we can add it to A, and we are done. So let us assume that no hyperedge is contained completely inside B'.

We use a counting argument to show that there must be many hyperedges with exactly $k-1$ elements in B'. Let a denote the number of such hyperedges e (satisfying $|e \cap B'| = k-1$), and b the number of hyperedges such that $|e \cap B'| \leq k-2$. All hyperedges in B fall into one of these two categories; hence $|B| = k(a+b)$. On the other hand, $|B'| \leq (k-1)a + (k-2)b$ which means that $|A| = |B| - |B'| \geq a + 2b$. We assumed that $|A| = (2/k - \epsilon)|B|$, which implies

$$a + 2b \leq |A| = (2/k - \epsilon)\,|B| = (2 - k\epsilon)\,(a+b). \tag{5.1}$$

We conclude that

$$a \geq k\epsilon(a+b). \tag{5.2}$$

Let Q denote the set of these a hyperedges in B, and $V(Q)$ denote the elements of B that belong to hyperedges in Q; each of them contains exactly one element in $B \setminus B'$ and $k-1$ elements in B'. Let $B_0 = V(Q) \cap B'$ and $B_1 = V(Q) \cap (B \setminus B')$. We have $|B_0| = (k-1)a$ and $|B_1| = a$.

Let $\mathcal{M}_0 = \mathcal{M}/B_0$ denote the matroid with B_0 contracted. Because $A \cup B_0 \subseteq A \cup B'$ and $B_1 \cup B_0 \subseteq B$, both of which are independent in $\mathcal{M}$, we get that A and B_1 are independent in $\mathcal{M}_0$. Because $|A| \geq a + 2b \geq |B_1|$, we can extend B_1 by adding (possibly zero) elements from A, to form an $\mathcal{M}_0$-independent set $(A \setminus A_1) \cup B_1$ where $|A_1| = |B_1| = a$.

If A contains any hyperedge e with $|e \cap A_1| \geq 2$, we find a local move of size two as follows: $((A \setminus e) \setminus A_1) \cup B_1$ is an independent set in $\mathcal{M}_0$, whose cardinality is at least $|A \setminus e| + 2$ (because A_1 contains ≥ 2 elements of e). Therefore, $A \setminus e$ can be extended to a set $(A \setminus e) \cup \{x', x''\}$ independent in $\mathcal{M}_0$, such that $x', x'' \in B_1$. The elements x', x'' are contained in hyperedges e', e'' respectively, such that $e' \setminus \{x'\}$ and $e'' \setminus \{x''\}$ are subsets of B_0. Because $(A \setminus e) \cup \{x', x''\}$ is independent in $\mathcal{M}_0 = \mathcal{M}/B_0$, any elements of B_0 can be added for free, and $(A \setminus e) \cup e' \cup e''$ is independent in $\mathcal{M}$. This defines a local move of size two.

The rest of the proof deals with the case in which there is no hyperedge in A with more than 1 element in A_1. Let P be the collection of hyperedges in A intersecting

A_1; each such hyperedge satisfies $|e \cap A_1| = 1$, and so $|P| = |A_1| = a$. Let A_2 denote the remaining elements of P, i.e., $A_2 \subseteq A \setminus A_1$ and $|A_2| = (k-1)a$. Here we apply the inductive hypothesis.

The inductive step. We define a new matroid $\mathcal{M}_1 = \mathcal{M}_0/B_1 = \mathcal{M}/(B_0 \cup B_1)$. By construction, the sets $A^* = A \setminus (A_1 \cup A_2)$ and $B^* = B \setminus (B_0 \cup B_1)$ are both independent in $\mathcal{M}_1$. They both form a union of hyperedges and hence feasible solutions to the matroid k-parity problem for $\mathcal{M}_1$. We have $|A^*| = |A| - ka$ and $|B^*| = |B| - ka = kb$. Using $|A| = (2 - k\epsilon)(a + b)$ from (5.1), we get

$$\frac{|A^*|}{|B^*|} = \frac{|A| - ka}{kb} = \frac{(2 - k\epsilon)(a + b) - ka}{kb} = \frac{2}{k} - \epsilon - \frac{(k - 2 + k\epsilon)a}{kb}$$

and applying (5.2) to estimate $a \geq kb\epsilon$, we get

$$\frac{|A^*|}{|B^*|} \leq \frac{2}{k} - \epsilon - (k - 2 + k\epsilon)\epsilon \leq \frac{2}{k} - (k-1)\epsilon.$$

Because we assumed $\epsilon > \frac{1}{(k-1)^t}$, we have

$$|A^*| < \left(\frac{2}{k} - \frac{1}{(k-1)^{t-1}}\right)|B^*|,$$

and we can apply the inductive hypothesis. There is a local move of size $s \leq (2k+1)^{t-2}$ between A^* and B^*, i.e., a union of $s-1$ hyperedges $\tilde{A} \subseteq A^*$ and s hyperedges $\tilde{B} \subseteq B^*$ such that $(A^* \setminus \tilde{A}) \cup \tilde{B}$ is independent in $\mathcal{M}_1$. Our goal is to find a local move of size at most $(2k+1)s$ between A and B (in $\mathcal{M}$).

We accomplish this by a construction essentially identical to the case of matroid parity. The set $(A^* \setminus \tilde{A}) \cup \tilde{B}$ is independent in $\mathcal{M}_1$. Unfortunately, $(A \setminus \tilde{A}) \cup \tilde{B}$ is not necessarily independent, even in $\mathcal{M}$. However, the set $(A^* \setminus \tilde{A}) \cup A_2 = A \setminus (A_1 \cup \tilde{A})$ is independent in $\mathcal{M}_1 = \mathcal{M}_0/B_1$, because $(A \setminus A_1) \cup B_1$ was constructed to be independent in $\mathcal{M}_0$. Therefore, we can extend $(A^* \setminus \tilde{A}) \cup \tilde{B}$ to a set $(A^* \setminus \tilde{A}) \cup \tilde{B} \cup (A_2 \setminus C_2)$ independent in $\mathcal{M}_1$, where $C_2 \subsetneq A_2$ and $|C_2| \leq |\tilde{B}|$. (As a special case, this contains the possibility that $|A_2| \leq |\tilde{B}|$, in which case we do not add any new elements and set $C_2 = A_2$.)

The new set $(A^* \setminus \tilde{A}) \cup \tilde{B} \cup (A_2 \setminus C_2)$ is also independent in $\mathcal{M}_0$ (a weaker condition). So is $(A^* \setminus \tilde{A}) \cup (A_2 \setminus C_2) \cup A_1$, as any subset of A is independent in $\mathcal{M}_0$. So, we can extend $(A^* \setminus \tilde{A}) \cup \tilde{B} \cup (A_2 \setminus C_2)$ to a set $(A^* \setminus \tilde{A}) \cup \tilde{B} \cup (A_2 \setminus C_2) \cup (A_1 \setminus C_1)$ independent in $\mathcal{M}_0$, where $C_1 \subseteq A_1$ and $|C_1| \leq |\tilde{B}|$.

The set we have obtained is not necessarily a union of hyperedges, so let us remove the entire hyperedge for each element in C_1 and C_2. Let us denote by C' the union of all hyperedges intersecting $C_1 \cup C_2$. Note that due to our construction and the fact that $A_1 \cup A_2$ is a disjoint union of hyperedges, $C_1 \cup C_2 \subseteq C' \subseteq A_1 \cup A_2$. We also define $C'_1 = C' \cap A_1$ and $C'_2 = C' \cap A_2$. We know that each hyperedge on $A_1 \cup A_2$ contains exactly one element in A_1 and $k-1$ elements in A_2. Therefore, $|C'_2| = (k-1)|C'_1|$, and also $|C'_1| = \frac{1}{k}|C'| \leq |C_1 \cup C_2|$, because each element of $C_1 \cup C_2$ contributes at most one hyperedge to C'.

We obtain a set $A^+ = (A^* \setminus \tilde{A}) \cup \tilde{B} \cup (A_2 \setminus C'_2) \cup (A_1 \setminus C'_1)$ in $\mathcal{M}_0$. Now, consider the set $(A^+ \setminus A_1) \cup B_1 = (A^* \setminus \tilde{A}) \cup \tilde{B} \cup (A_2 \setminus C'_2) \cup B_1$. This is independent in $\mathcal{M}_0$, because $A^+ \setminus A_1 = (A^* \setminus \tilde{A}) \cup \tilde{B} \cup (A_2 \setminus C'_2) \subseteq (A^* \setminus \tilde{A}) \cup \tilde{B} \cup (A_2 \setminus C_2)$ was constructed to be independent in $\mathcal{M}_1 = \mathcal{M}_0/B_1$. If A^+ misses some elements in A_1, namely if $C'_1 \neq \emptyset$, the cardinality of $(A^+ \setminus A_1) \cup B_1$ is actually larger than $|A^+|$, namely $|(A^+ \setminus A_1) \cup B_1| = |A^+| + |C'_1|$. Hence, we can extend A^+ by $F_1 \subseteq B_1$, $|F_1| = |C'_1|$, to obtain a set $A^+ \cup F_1$ independent in $\mathcal{M}_0$. (If $C'_1 = \emptyset$, we just continue with $F_1 = \emptyset$.) Each hyperedge touching F_1 has exactly 1 element in B_1 and the remaining $k-1$ elements in B_0; denote the elements of these hyperedges in B_0 by F_0. As $A^+ \cup F_1 \in \mathcal{M}_0 = \mathcal{M}/B_0$, we can add F_0 and obtain an independent set $A^+ \cup F_1 \cup F_0$ in $\mathcal{M}$. We have $|F_1| = |C'_1|$ and $|F_0| = (k-1)|F_1| = (k-1)|C'_1| = |C'_2|$. To conclude, we have found a feasible solution $A^+ \cup F_1 \cup F_0$ in $\mathcal{M}$, of cardinality

$$|A^+ \cup F_1 \cup F_0| = |A^+ \cup C'_1 \cup C'_2| = |(A \setminus \tilde{A}) \cup \tilde{B}| > |A|.$$

Finally, let us estimate the size of this local move. The local move removes $\tilde{A} \cup C'_1 \cup C'_2$ from A, and adds $\tilde{B} \cup F_0 \cup F_1$ instead. As we showed above, the size of C'_1 is bounded by $|C'_1| \leq |C_1 \cup C_2| \leq 2|\tilde{B}|$, hence $|F_0 \cup F_1| = k|C'_1| \leq 2k|\tilde{B}|$. In summary, we are adding at most $(2k+1)|\tilde{B}|$ elements to A, i.e., the size of the local move is at most $(2k+1)|\tilde{B}| \leq (2k+1)^{t-1}$. □

6. The weighted matchoid problem. In this section we analyze the integrality gap for the *weighted matchoid* problem. That is, given an edge-weighted graph $G = (V, E)$ and a matroid $\mathcal{M}_v = (\delta(v), \mathcal{I}_v)$ on the edges incident to v for each $v \in V$, find the maximum-weight subset of edges F such that $F \cap \delta(v)$ is independent in matroid $\mathcal{M}_v$ for each $v \in V$. This is a special case of matroid matching as we discussed in Section 1.

First, we show a result similar to the one in [22].

LEMMA 6.1. *The linear-programming relaxation (4.1–4.4) for the weighted matroid matching problem ($k = 2$) has a half-integral optimal solution, i.e., a solution x^*, y^* such that $x^*_u, y^*_e \in \{0, \frac{1}{2}, 1\}$. Moreover, there exists a polynomial time algorithm to find such a solution.*

Proof. Consider a basic optimal solution (x^*, y^*) of the linear-programming relaxation (4.1–4.4). We can assume that the edges $e \in \mathcal{E}$ form a matching, i.e., are pairwise disjoint. Let $\mathcal{T}$ be the collection of all sets such that their corresponding inequalities (4.2) are saturated, i.e., satisfied with equalities. By the properties of basic solutions of linear programs, the solution (x^*, y^*) is a unique solution of the system of linear equations corresponding to these equalities, equations (4.3) and constraints fixing some variables to have zero value, i.e., saturated constraints (4.4). Because the equations (4.3) are just equalities between different variables, we can replace the variables x_u by y_e where e is the unique edge such that $u \in e$. This leaves $|\mathcal{E}| = |V|/2$ independent variables y_e. The submodularity of the rank function implies that if $A, B \in \mathcal{T}$ then the constraints corresponding to the sets $A \cap B$ and $A \cup B$ are also saturated, i.e., $A \cap B \in \mathcal{T}$ and $A \cup B \in \mathcal{T}$.

Consider any maximal chain $\emptyset = T_0 \subset T_1 \subset \cdots \subset T_k$ such that each $T_i \in \mathcal{T}$. Because we chose a maximal chain, any set $S \in \mathcal{T}$ must be contained in T_k; otherwise, we can extend our chain by adding another tight set $T_{k+1} = T_k \cup S$. Also, for every $i = 0, \ldots, k-1$, we have $S \cup T_i \in \mathcal{T}$ and $(S \cup T_i) \cap T_{i+1} \in \mathcal{T}$. Therefore, if $S \cap (T_{i+1} \setminus T_i) \neq \emptyset$ and $T_{i+1} \setminus T_i \not\subseteq S$ for some $i = 0, \ldots, k-1$, then we can extend our chain by inserting the set $(S \cup T_i) \cap T_{i+1} \in \mathcal{T}$ between T_i and T_{i+1} which contradicts the maximality of our chain.

To summarize, for any set $S \in \mathcal{T}$ we have $S \subseteq T_k$ and for every $i = 0, \ldots, k-1$, either $S \cap (T_{i+1} \setminus T_i) = \emptyset$ or $T_{i+1} \setminus T_i \subseteq S$. This implies that the saturated constraint corresponding to S can be represented as a linear combination of the saturated constraints (4.2) corresponding to the chain $T_0 \subset T_1 \subset \cdots \subset T_k$. These constraints can be rewritten as $\sum_{u \in T_{i+1} \setminus T_i} x_u = r_{\mathcal{M}}(T_{i+1}) - r_{\mathcal{M}}(T_i)$ for each $i = 0, \ldots, k-1$.

This new set of constraints has the crucial property that each variable x_u such that $x^*_u > 0$ appears in exactly one of the above constraints. Together with the constraints $x_u = x_v$ for $(u,v) \in \mathcal{E}$, we obtain an instance of the classical b-matching problem. We have a vertex corresponding to each $i = 0, \ldots, k-1$, $b_i = r_{\mathcal{M}}(T_{i+1}) - r_{\mathcal{M}}(T_i)$ and edges correspond to pairs in the matroid matching instance (or constraints (4.3)). It is known that the b-matching polyhedron (defined by constraints $y_e \geq 0$ and $\sum_{e \in \delta(v)} y_e = b_v$ for each $v \in V$) is half-integral [2]. I.e., if we have variables defined on edges $y_e > 0$ and we know that this solution is a unique solution of the system of linear equations defined by the degree constraints in the b-matching problem, then this solution must be half-integral and can be found by standard linear programming techniques. □

Let LP^* be the optimal value of the linear program (4.1–4.4). Using Lemma 6.1 we prove the following.

THEOREM 6.2. *There exists a polynomial time algorithm that finds a feasible solution to the weighted matchoid problem of value at least $\frac{2}{3}LP^*$, given the half-integral optimal solution of the linear-programming relaxation (4.1–4.4).*

Proof. Recall the reduction of the matchoid problem to the matroid parity problem from Introduction. In this reduction each edge (u,v) in the matchoid problem corresponds to a pair that includes its own copies of u and v. Each vertex v in the matchoid instance has its own matroid $\mathcal{M}_v$ defined on edges incident to that vertex, or equivalently on copies of vertex v. The key property of the matchoid problem (or matroid parity instance corresponding to the matchoid problem) that allows us to derive an approximation algorithm is that each rank constraint involves at most one element from each pair. This is because each rank constraint comes from a matroid that is defined on the edges incident to a fixed vertex of the original instance.

Let (x^*, y^*) be the optimal half-integral solution of the linear-programming relaxation (4.1–4.4) defined on the instance of the matroid parity problem corresponding to the matchoid problem. Recall, that x-variables correspond to the elements of the ground sets in matroids $\mathcal{M}_v$ for $v \in V$, i.e. there are variables $x_{u,e}$ and $x_{v,e}$ for each edge $e = (u,v)$. The y-variables correspond to the edges in the underlying graph $G = (V, E)$. For each edge $e = (u,v)$, we will associate a pair (u,e) with each element of matroid $\mathcal{M}_v$ for $v \in V$.

Our algorithm rounds variables iteratively. In iteration i we are given an instance of the matroid parity problem corresponding to the matchoid problem. In this instance we have matroids $\mathcal{M}_{v,i}$ for each vertex $v \in V$ of the original matchoid instance and we are given a set of pairs $\mathcal{E}_i$ such that the elements of the pair always participate in the different matroid constraints. We are also given a half-integral solution (x^i, y^i) of the linear program (4.1–4.4) for that instance. Initially, $\mathcal{E}_0 = \mathcal{E}$, $\mathcal{M}_{v,0} = \mathcal{M}_v$ for each vertex $v \in V$ of the matchoid instance and $(x^0, y^0) = (x^*, y^*)$.

In iteration i our algorithm chooses a pair $e_h = (u,v) \in \mathcal{E}_i$ such that $y^i_{e_h} > 0$ of highest weight w_h. We add this pair to our current approximate solution and update matroids $\mathcal{M}_{v,i+1} = \mathcal{M}_{v,i}/e_h$ and $\mathcal{M}_{u,i+1} = \mathcal{M}_{u,i}/e_h$, i.e. we consider contraction of matroids by the element e_h. For all other vertices $q \in V$, we define $\mathcal{M}_{q,i+1} = \mathcal{M}_{q,i}$. If $y^i_{e_h} = 1$ then we do not need to change the fractional solution. In this case $x^{i+1}_{(q,e)} = x^i_{(q,e)}$ and $y^{i+1}_{e_{h'}} = y^i_{e_{h'}}$ for all $q \neq u, v$ and $h' \neq h$. Otherwise, $y^i_{e_h} = 1/2$. In this case the old solution may not be feasible anymore for rank inequalities defined by matroids $\mathcal{M}_{u,i+1}$ and $\mathcal{M}_{v,i+1}$.

Note that x^i is a half-integral feasible solution of the matroid polyhedron $\mathcal{P}(\vee_{q \in V} \mathcal{M}_{q,i})$ (rank constraints (4.2)) where $\vee_{q \in V} \mathcal{M}_{w,i}$ the union of matroids $\mathcal{M}_{q,i}$. By the integer decomposition property [48] (Corollary 42.1e, p.730) this vector is a convex combination of two independent sets I_1 and I_2 in this matroid. Each of those independent sets consists of a union of independent sets in matroids $\mathcal{M}_{q,i}$, i.e., $I_1 = \cup_{q\in V} I_{q,1}$ and $I_2 = \cup_{q\in V} I_{q,2}$ where $I_{q,1}, I_{q,2} \in \mathcal{I}(\mathcal{M}_{q,i})$.

Because $y^i_{e_h} = x^i_{(u,e)} = x^i_{(v,e)} = 1/2$, we know that the pair (u, e_h) belongs to either $I_{u,1}$ or $I_{u,2}$. Without loss of generality, assume $(u, e_h) \in I_{u,1}$. By the matroid exchange properties there exists an element $\pi((u, e_h)) = (u, e_{h'})$ such that the set $I_{u,2} \cup \{(u, e_h)\} \setminus \{\pi((u, e_h))\} \in \mathcal{I}(\mathcal{M}_{u,i})$, i.e., it is independent in this matroid.

For the element $\pi((u, e_h)) = (u, e_{h'})$, let $e_{h'} = (u, q')$ for some vertex $q' \in V$. If $y^i_{e_{h'}} = x^i_{(u,e_{h'})} = x^i_{(q',e_{h'})} = 1/2$ then we define $y^{i+1}_{e_{h'}} = x^{i+1}_{(u,e_{h'})} = x^{i+1}_{(q',e_{h'})} = 0$. If $y^i_{e_{h'}} = x^i_{(u,e_{h'})} = x^i_{(q',e_{h'})} = 1$ then $y^{i+1}_{e_{h'}} = x^{i+1}_{(u,e_{h'})} = x^{i+1}_{(q',e_{h'})} = 1/2$.

We apply an analogous operation for the element $(v, e_h) \in \mathcal{M}_{v,i}$, i.e., we define an element $\pi((v, e_h)) = (v, q'')$ and pair $e_{h''} = (v, q'')$ and update the variables accordingly. We do not change any other variables, i.e., $y^{i+1}_{e_g} = y^i_{e_g}$ for $g \neq h, h', h''$ and $x^{i+1}_{(q,g)} = x^i_{(q,g)}$ for

$$(q, g) \neq (u, e_h), (v, e_h), (u, e_{h'}), (v, e_{h''}), (q', e_{h'}), (q'', e_{h''}).$$

We claim that the new fractional solution (x^{i+1}, y^{i+1}) is feasible for the matroid matching problem for the next iteration. The key property we used is that the elements from the pair (u, v) participate in different matroid constraints and do not appear in the same independent set $I_{u,i}$ or $I_{v,i}$ for $i = 1, 2$.

In each iteration i we added a pair of value w_h to our approximate solution and decreased the value of the LP solution for the next iteration by at most $\frac{3}{2}w_h$. It implies that our final approximate solution will have value at least $\frac{2}{3}LP^*$. □

We remark that the integrality gap of LP (4.1–4.4) is 3/2 even for the special case of non-bipartite matching (the example is a triangle), so with respect to this LP we cannot achieve a better approximation.

7. Conclusion. We have seen that a simple combinatorial algorithm performs dramatically better than any known LP-based approach for matroid matching. Linear programming still holds some promise for the k-uniform matchoid problem. Our $\frac{3}{2}$-approximation for the weighted matchoid problem and the results from [9] on the integrality gap of the hypergraph matching problem motivate the following.

CONJECTURE 1. *The integrality gap of the LP relaxation (4.1–4.4) is $k - 1 + \frac{1}{k}$ for the maximum weighted k-matchoid problem and $k - 1$ for the maximum weighted k-matroid intersection problem.*

In the case of weighted matroid k-parity, we have the following conjecture, which is true (and tight) for the weighted k-set packing problem due to [1, 3] and also for the weighted k-matroid intersection problem due to [34].

CONJECTURE 2. *The simple local-search algorithm for the weighted matroid k-parity problem that tries to add/remove a constant number of hyperedges in each iteration has approximation guarantee $k - 1 + \varepsilon$ for any $\varepsilon > 0$ (with running time depending on $1/\varepsilon$).*

A very intriguing open problem is to show that this simple local-search algorithm gives a PTAS for weighted matroid parity ($k = 2$). This would be the first PTAS in the weighted case for general matroids; we remark that even for linearly-represented matroids, there is only a randomized FPTAS which follows from known pseudopolynomial-time randomized algorithms [8, 40].

Another interesting line of research is to analyze more sophisticated local-search algorithms (see [5, 10]) implemented for the weighted matroid k-parity problem. Such algorithms are known to provide improved approximation guarantees for the weighted set packing problem.

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