Randomized rounding in matroid polytopes

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6 problems

What do the following problems have in common?

- **Degree-bounded Spanning Tree**
- **Geometric Stabbing Spanning Tree**
- **Welfare Maximization in Combinatorial Auctions**
- **Max-Min Allocation Problem**
- **Generalized Assignment Problem**
- **Asymmetric Traveling Salesman Problem**

Matroids

Definition

A matroid on N is defined by a system of *bases* $\mathcal{B} \subset 2^N$, satisfying

1. $\forall B_1, B_2 \in \mathcal{B}; |B_1| = |B_2|$.
2. $\forall B_1, B_2 \in \mathcal{B}; \forall x \in B_1; \exists y \in B_2;$
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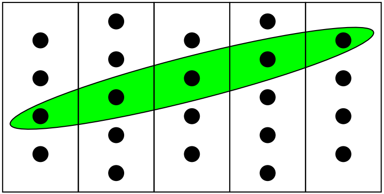
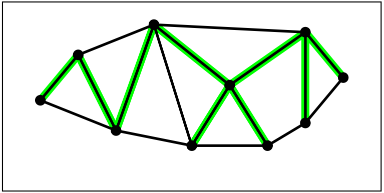
Examples:



partition matroid

(bases = transversals)



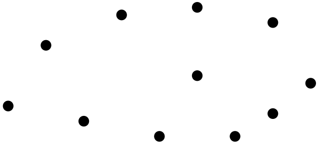
graphic matroid

(bases = spanning trees)

Minimum Stabbing Spanning Tree

Given n points $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n \in \mathbb{R}^d$, find a spanning tree T such that no hyperplane cuts too many edges of T.

Def.: *stabbing number* $\sigma(T)$ = maximum number of edges of T intersected by any hyperplane.



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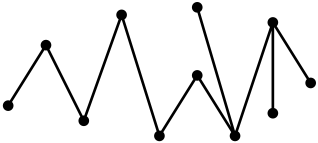

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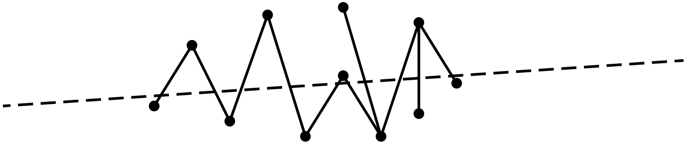

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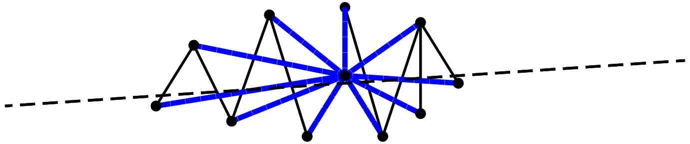

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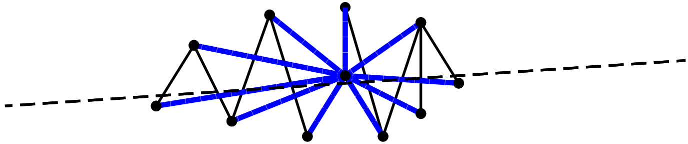
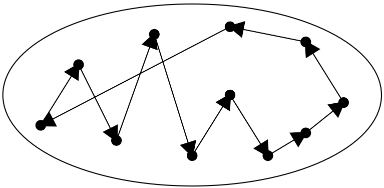


Known: $O(\log n)$-approximation for finding a spanning tree minimizing the stabbing number $\sigma(T)$ [Har-Peled '08].

Asymmetric Traveling Salesman Problem

ATSP: Given a directed graph with edge costs c_e, find a directed circuit of minimum cost, visiting all vertices.



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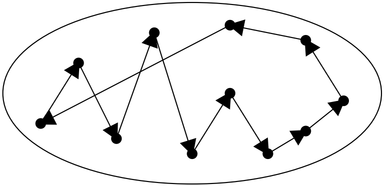

Asadpour, Goemans, Madry, Oveis-Gharan, Saberi '10:
$O(\log n / \log\log n)$-approximation, improving the previously long-standing $O(\log n)$-approximation.

Key ingredient: Given a fractional spanning tree x, find a spanning tree T which does not contain too many edges across any cut $(S, \overline{S})$:

$$|T \cap e(S, \overline{S})| \leq \alpha x(S, \overline{S}).$$

Allocation problems

Social Welfare Maximization: *Given n agents with valuation functions $w_i : 2^X \to \mathbb{R}_+$. Allocate items to maximize $\sum_{i=1}^{n} w_i(S_i)$.*

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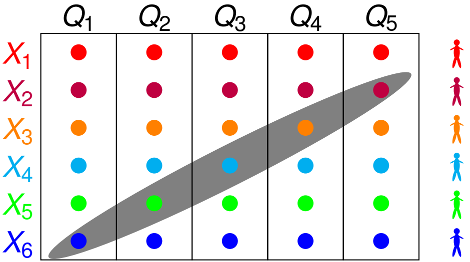

Reduction:

Create n clones of each item,
$f(S) = \sum w_i(S \cap X_i)$,
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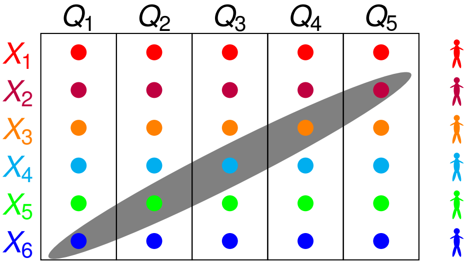

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Known: $(1 - 1/e)$-approximation for submodular functions [V. '08]

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A different objective (a.k.a. Santa Claus problem):
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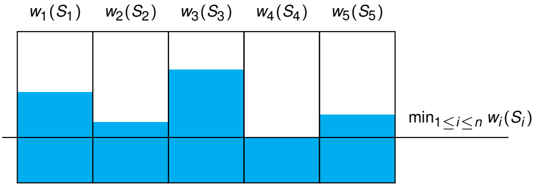

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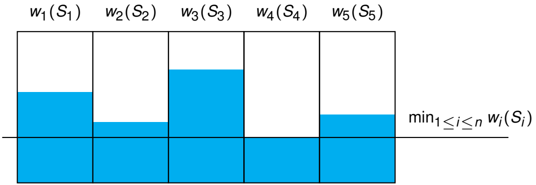

- much harder to approximate, even for linear valuation functions
- for constant n, there is an FPTAS [Woeginger 2000]
- for growing n, best known: n^ϵ-approximation in $O(n^{1/\epsilon})$ time [Chakrabarty, Chuzhoy, Khanna '09]
- for submodular valuations, $(2n-1)$-approximation [Khot, Ponnuswami '07]

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- Main issue in this talk: How to round a fractional solution?

Randomized rounding

Raghavan-Thompson '87: **randomized rounding.**

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However, what if we want to satisfy some constraints *exactly*?

$$\max\{\sum_i c_i x_i : Ax \leq b; x \geq 0; \sum x_i = k\}$$

Randomized rounding in matroid polytopes

Goal: Given a fractional solution $x \in B(\mathcal{M})$,
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1. $\forall i; \Pr[i \in R] = x_i$
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- *Maximum entropy sampling:* for x in the spanning tree polytope, concentration for arbitrary subsets [Asadpour et al. '09]

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We present a very simple rounding technique such that:

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- Marginals are preserved: $\Pr[i \in R] = x_i$ for all i.
- The events $i \in R$ are negatively correlated
  $\longrightarrow$ concentration for linear functions $a(R)$.
- We also prove a Chernoff-type tail estimate
  for *monotone submodular* functions $f(R)$.

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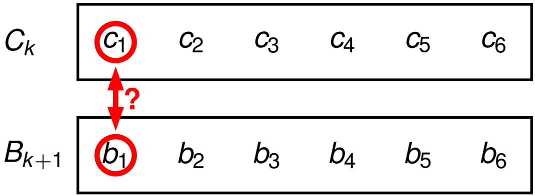

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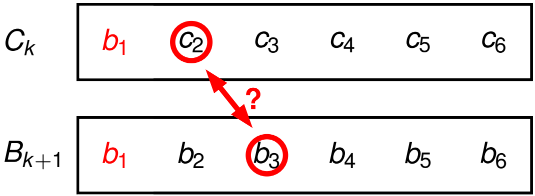

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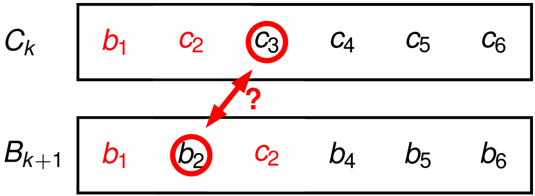

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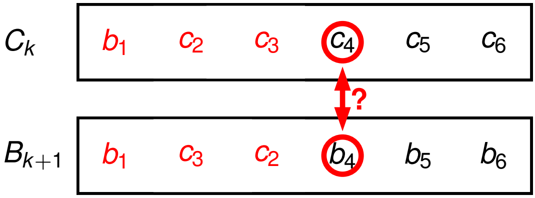

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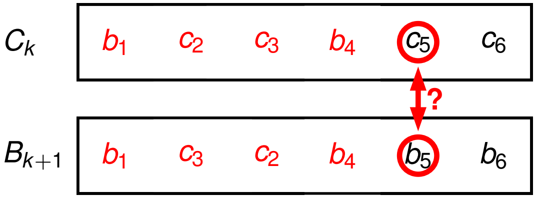

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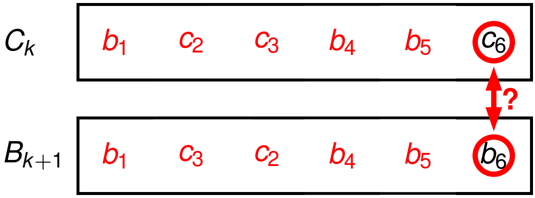

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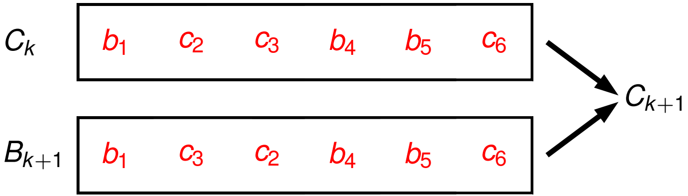

Properties of randomized swap rounding

Theorem (negative correlation)

Let X_i be the indicator variables for the events $i \in R$, where $R \in \mathcal{B}$ is obtained by randomized swap rounding from $x \in B(\mathcal{M})$. Then $\mathbb{E}[X_i] = x_i$ and for any $J \subseteq [n]$,

1. $\mathbb{E}[\prod_{j \in J} X_i] \leq \prod_{j \in J} x_i$.
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Using [Panconesi-Srinivasan '97], this implies Chernoff-type bounds.

Corollary

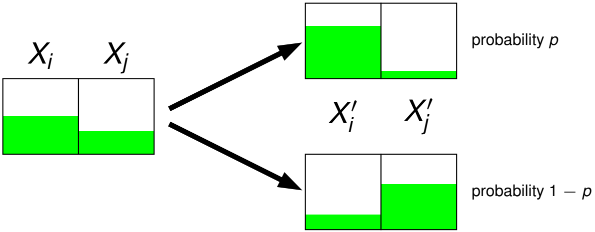
Let $a_1, \ldots, a_n \in [0,1]$ and $\mu = \sum a_i x_i$. Then for any $\delta > 0$

1. $\Pr[\sum a_i X_i < (1 - \delta)\mu] < e^{-\mu\delta^2/2}$.
2. $\Pr[\sum a_i X_i > (1 + \delta)\mu] < (e^\delta/(1+\delta)^{1+\delta})^\mu$.

Proof of negative correlation

Consider a step where X_i, X_j are modified into X_i', X_j'.

- $\mathbb{E}[X_i' \mid X_i] = X_i$.
- $X_i' + X_j' = X_i + X_j$.



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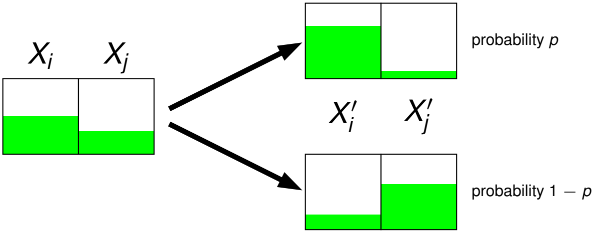

$$\begin{aligned} \mathbb{E}[X_i' X_j' \mid X_i, X_j] &= \frac{1}{4}\mathbb{E}[(X_i' + X_j')^2 \mid X_i, X_j] - \frac{1}{4}\mathbb{E}[(X_i' - X_j')^2 \mid X_i, X_j] \\ &\leq \frac{1}{4}(X_i + X_j)^2 - \frac{1}{4}(X_i - X_j)^2 = X_i X_j. \end{aligned}$$

Application 1

Minimum stabbing spanning tree: Find a geometric spanning tree on $x_1, \ldots, x_n \in \mathbb{R}^d$ such that no hyperplane intersects too many edges ($d =$ constant).

Result: $O(\log n / \log \log n)$-approximation for minimizing the stabbing number $\sigma(T)$ over all spanning trees.

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Minimum stabbing spanning tree: Find a geometric spanning tree on $x_1, \ldots, x_n \in \mathbb{R}^d$ such that no hyperplane intersects too many edges ($d =$ constant).

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- This improves a recent $O(\log n)$-approximation [Har-Peled '08].
- We write an LP using the graphic matroid $\mathcal{M}$, including constraints for the $O(n^d)$ combinatorially different hyperplanes:

$$\min\{\sigma : \exists x \in B(\mathcal{M}), Ax \leq \sigma \mathbf{1}\}.$$

- A fractional solution is rounded using randomized swap rounding; Chernoff bounds imply that every hyperplane intersects at most $O(\frac{\log n}{\log\log n})\sigma$ edges.

Application 2

Asymmetric Traveling Salesman: an alternative (simpler) way to derive the $O(\log n / \log\log n)$-approximation of Asadpour et al.

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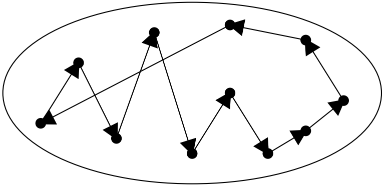

- Solve the natural LP: the fractional solution x^* can be interpreted as a point in the spanning tree polytope.
- We apply randomized swap rounding to x^*, to obtain a random spanning tree T.
- Chernoff bounds imply that for any cut C, $|T \cap C| \leq \frac{\log n}{\log\log n} x^*(C)$.
- Such a spanning tree can be extended to a salesman tour of cost $O(\frac{\log n}{\log\log n})OPT$, using the techniques of Asadpour et al.

Submodular functions

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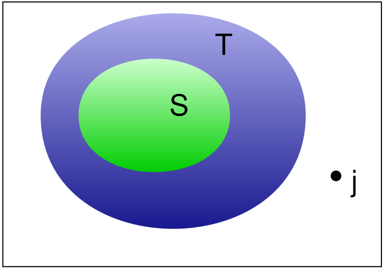
- **Submodularity:** Let the *marginal value* of element j be

$$f_S(j) = f(S+j) - f(S).$$



f is submodular, if
j adds more value to S than T:

$$f_T(j) \leq f_S(j)$$

Tail estimate for submodular functions

Theorem

Let $f : \{0,1\}^n \to \mathbb{R}_+$ be submodular with marginal values in $[0,1]$ and $X_1, \ldots, X_n \in \{0,1\}$ obtained by randomized swap rounding from $x \in B(\mathcal{M})$. Let $\delta \in [0,1]$ and $\mu = \mathbb{E}[F(x)]$ where F is the multilinear extension of f. Then $\mathbb{E}[f(X_1, \ldots, X_n)] \geq \mu$ and

- $\Pr[f(X_1, \ldots, X_n) < (1-\delta)\mu] < e^{-\mu\delta^2/8}$.

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- $\Pr[f(X_1, \ldots, X_n) < (1-\delta)\mu] < e^{-\mu\delta^2/8}$.

Proof: bounding the exponential moment $\mathbb{E}[e^{\lambda(\mu - f(R))}]$ by careful aggregation of contributions from successive steps; don't know how to use negative correlation.

Note: weaker results follow directly from martingale concentration bounds; due to the length of the rounding process, it is crucial to remove dependence on n.

Application 3

Max-Min Submodular Allocation. Given n agents with submodular valuations $w_i : 2^N \to \mathbb{R}_+$, find an allocation $(S_1, \ldots, S_n)$ maximizing $\min_{1 \le i \le n} w_i(S_i)$.

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Result: A $(1 - 1/e - \epsilon)$-approximation for any constant $n \ge 2$ (or assuming that all singleton values are $O(OPT/\log n)$).

- We solve the multilinear relaxation using a multiobjective variant of the continuous greedy algorithm.
- We use randomized swap rounding to generate a random allocation.
- The lower-tail bound for submodular functions is crucial in getting a high-probability guarantee for all agents simultaneously.

Open problems

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- Can randomized rounding techniques with negative correlation properties be extended to other structures than matroids? [very recently: *matroid intersection*]
- Can the rounding procedure be modified to be more sensitive to additional linear constraints?