Approximation Algorithms - Tools and Techniques **Spring 2002**

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Linear Program Rounding

Introduction

The idea of Linear Programming is based on the following theorem:

Theorem 1 *The maximum or minimum value of a linear function, subject to a set of linear constraints, can be obtained in polynomial time.*

Vertex Cover

The Vertex Cover problem is as follows. Given a graph $G(V, E)$, find the smallest set $V' \subset V$ such that all the edges in E are incident on some vertex in V'. This problem can be formulated as follows:

For every vertex v, $x(v) = \begin{cases} 0 & \text{if } v \notin V' \\ 1 & \text{if } v \in V' \end{cases}$

$$\min \sum x(v)$$

s.t.

$$\begin{aligned} x(v_1) + x(v_2) &\geq 1 && \forall e = (v_1, v_2) \\ x(v) &\in \{0, 1\} && \forall v \in V \\ \Rightarrow x(v)(1 - x(v)) &= 0 && \forall v \in V \end{aligned}$$

But the last constraint is not a linear constraint. The above set of constraints and objective function form an Integer Program problem, which can not be solved in polynomial time. So

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the last constraint is relaxed to obtain a Linear Program. The linear relaxation of the last constraint results in the new constraint:

$$0 \leq x(v) \leq 1$$

The above LP can be solved in polynomial time. But the solution obtained for the variables $x(v)$ may be fractional, which doesn't have any significance in the original Vertex Cover problem. An example of the fractional solution to the vertex cover problem is shown in Figure 1

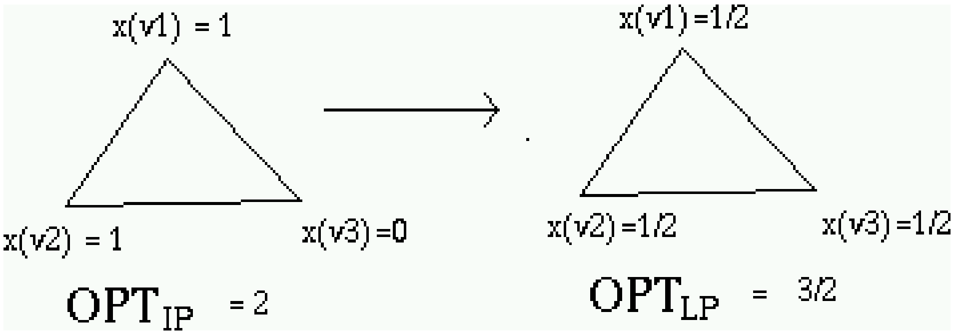

Figure 1: LP rounding for K_3 graph

To obtain an integer solution from the fractional solution, *rounding* or *massaging* is applied to the fractional solution. The goal of LP rounding is to convert the optimal LP solution to an approximate integer solution.

The optimal integer solution OPT_{IP} is a feasible solution to the LP

$$\begin{aligned}\Rightarrow\ OPT_{IP} &\geq OPT_{LP} \\ \Rightarrow\ OPT_{LP} &= \text{A lower bound for optimal solution to the vertex cover}\end{aligned}$$

Goal of LP rounding

From the optimal solution OPT_{LP}, the $x(v)$ values are massaged so as to get integral values for all the variables. Let the cost of this solution be C_{ROUND}.

$$\begin{aligned}\text{If } C_{ROUND} &\leq \alpha OPT_{LP} \\ &\leq \alpha OPT_{IP} \\ \Rightarrow \frac{C_{ROUND}}{OPT_{IP}} &\leq \alpha\end{aligned}$$

and hence gives an α-approximation to the problem.

For the vertex cover problem,

$$\begin{array}{rcll} x(v_1)+x(v_2) & \geq & 1 & \forall e=(v_1,v_2) \\ \Rightarrow \max\{x(v_1),x(v_2)\} & \geq & 1/2 & \forall e=(v_1,v_2) \end{array}$$

So a rounding scheme is to set

$$\tilde{x}(v)=\begin{cases} 1 \text{ if } x(v)\geq 1/2 \\ 0 \text{ if } x(v)<1/2 \end{cases} \tag{1}$$

The above equations imply:

$$\tilde{x}(v_1)+\tilde{x}(v_2)\geq 1 \forall e=(v_1,v_2)$$

This implies $\tilde{x}(v)$ is a valid solution to the IP.

$$\begin{array}{rcl} OPT_{LP} & = & \sum_v x(v) \\ C_{ROUND} & = & \sum_v \tilde{x}(v) \end{array}$$

But,

$$\begin{array}{rrcl} & \tilde{x}(v) & \leq & 2\cdot x(v) \\ \Rightarrow & \sum_v \tilde{x}(v) & \leq & 2\sum_v x(v) \\ \Rightarrow & C_{ROUND} & \leq & 2OPT_{LP} \end{array}$$

Therefore, Vertex Cover has a 2 approximation.

The question is whether this bound is tight. It can be shown that there cannot be much better approximation ratios than this for vertex cover. Consider the case of K_3.

$$\frac{OPT_{IP}}{OPT_{LP}}=\frac{2}{3/2}=4/3$$

This shows that the approximation ratio is $\geq 4/3$. The notion of *Integrality Gap* gives a bound on approximation ratios obtainable by the LP rounding method. Let I denote the set of all problem instances. Then Integrality gap α is given by

$$\alpha=\max_{i\in I}\frac{OPT_{IP}(i)}{OPT_{LP}(i)}$$

It is clear that we cannot obtain a better than α-approximation if we compare the rounded solution with the LP solution OPT_{LP}

Theorem 2 *Vertex cover has an integrality gap* $\alpha=2-\frac{2}{|V|}$

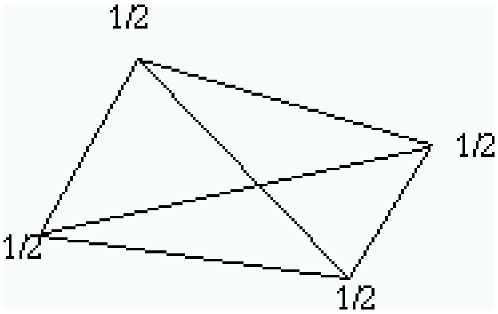

Figure 2: LP rounding for K_3

For K_4 (Figure 2),

$$\begin{aligned} OPT_{LP} &= 4 \cdot 1/2 = 2 \\ OPT_{IP} &= 3 \\ \Rightarrow \frac{OPT_{IP}}{OPT_{LP}} & = 3/2 \end{aligned}$$

For K_n,

$$\begin{aligned} OPT_{LP} &= n/2 \\ OPT_{IP} &= n-1 \\ \Rightarrow \quad \frac{OPT_{IP}}{OPT_{LP}} &= (n-1) \cdot \frac{2}{n} = 2 - \frac{2}{n} \end{aligned}$$

Weighted Vertex Cover

The Integer programming for the weighted vertex cover problem is

$$\min \sum_{v \in V} w(v) \cdot x(v)$$

$$\begin{aligned} x(v_1) + x(v_2) &\geq 1 \qquad \forall e = (v_1, v_2) \\ x(v) &\in \{0,1\} \quad \forall vinV \end{aligned}$$

The previous rounding scheme used in the unit weight vertex cover problem gives a 2 approximation for the weighted vertex cover problem.

The LP Solution for MAX-CUT

The LP rounding method need not give meaningful solutions to many approximation problems. For example, the LP rounding scheme gives a useless solution to the MAX-CUT problem. The

MAX-CUT problem is to find a partition (V_1, V_2) of vertices such that the number of edges from V_1 to V_2 is maximized. This problem can be formulated as follows:

$$\begin{aligned} x(v) &= \begin{cases} 0 \text{ if } v \in V_1 \\ 1 \text{ if } v \in V_2 \end{cases} \\ y(e) &= \begin{cases} 1 \text{ if the edge } e \text{ crosses the cut} \\ 0 \text{ otherwise} \end{cases} \\ \max \sum_{e \in E} y(e) & \end{aligned}$$

The relation between the variables $x(v)$ and $y(e)$ is given by the following table, where $e = (v_1, v_2)$:

$x(v_1)$	$x(v_2)$	$y(e)$
0	0	0
0	1	1
1	0	1
1	1	0

This relationship can be expressed by the constraint

$$y(e) = x(v_1) \oplus x(v_2)$$

The above is not a linear constraint. So, it is rephrased as:

$$\begin{array}{rrcl} & y(e) & = & \min\{x(v_1) + x(v_2), 2 - x(v_1)) - x(v_2)\} \\ \Rightarrow & y(e) & \leq & x(v_1) + x(v_2) \\ \text{and} & y(e) & \leq & 2 - x(v_1) - x(v_2) \\ & x(v) & \in & [0, 1] \end{array}$$

The OPT_{LP} will be

$$\begin{array}{rrcll} & x(v) & = & \frac{1}{2} & \forall v \in V \\ & y(e) & = & 1 & \forall e \in E \\ \Rightarrow & \sum_e y(e) & = & |E| & \end{array}$$

The LP solution had given the complete edge set in the cut. Any rounding scheme does not get benefited from the LP solution as all the variables are set to the same value. Hence the solution is a useless one.

Chernoff Bounds

Chernoff bound gives an upper bound on the probability that a sum of independent random variables is above a particular value.

Theorem 3 *Let $X_1, X_2, \ldots X_n$ be independent random variables. with*

$$X_i = \begin{cases} 1 \text{ with probability } p_i \\ 0 \text{ with probability } 1 - p_i \end{cases}$$

Then, $\forall \alpha > 0, t > 0$

$$\begin{aligned} Pr[\textstyle\sum_{i=1}^k X_i > t] &\leq e^{-\alpha t} \textstyle\prod_{i=1}^k E[e^{\alpha X_i}] \\ &= e^{-\alpha t} \textstyle\prod_{i=1}^k (p_i e^\alpha + (1 - p_i)) \end{aligned}$$

Proof:

$$Pr[\sum_{i=1}^k X_i > t] = Pr[e^{\sum_{i=1}^k X_i} > e^{\alpha t}]$$

Applying Markov's inequality,

$$\begin{aligned} Pr[\textstyle\sum_{i=1}^k X_i > t] &\leq \frac{E[e^{\sum_{i=1}^k X_i}]}{e^{\alpha t}} \\ &= \frac{E[\prod_{i=1}^k e^{\alpha X_i}]}{e^{\alpha t}} \end{aligned}$$

Since the random variables represent Bernoulli trials,

$$E[\prod_{i=1}^k e^{\alpha X_i}] = e^\alpha p_i + (1 - p + i)$$

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Corollary 1 *Let $M = \sum_{i=1}^k p_i$. For $\beta > 0$, set:*

$$\begin{aligned} \alpha &= \ln(1 + \beta) \\ t &= E[\textstyle\sum_{i=1}^k X_i](1 + \beta) \\ &= (1 + \beta) \textstyle\sum_{i=1}^k p_i \\ &= (1 + \beta) M \end{aligned}$$

Then,

$$\begin{aligned} Pr[\textstyle\sum_{i=1}^k X_i > (1 + \beta) M] &\leq \left(\frac{e^\beta}{(1+\beta)^{(1+\beta)}} \right)^M \\ &\approx \left(\frac{\beta}{e} \right)^{-\beta M} \end{aligned}$$

Proof:

$$
\begin{array}{rcl}
Pr[\sum_{i=1}^{k} X_i > (1+\beta)M] & & \\
& \leq & e^{-\ln(1+\beta)(1+\beta)M} \prod_{i=1}^{k}[p_i e^{\ln(1+\beta)} + 1 - p_i] \\
& = & \frac{1}{(1+\beta)^{(1+\beta)M}} \cdot \prod_{i=1}^{k}(p_i\beta + 1) \\
& \leq & \frac{1}{(1+\beta)^{(1+\beta)M}} \cdot \prod_{i=1}^{k} e^{\beta p_i} \\
& = & \frac{1}{(1+\beta)^{(1+\beta)M}} \cdot e^{\beta M} \\
& = & \left(\frac{e^{\beta}}{(1+\beta)^{(1+\beta)}}\right)^{M}
\end{array}
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