The Relation Between the Path of Centers and Smale's Regularization of the Linear Programming Problem

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ABSTRACT

Smale proposed a framework for applying Newton's method to the linear programming problem. It is shown that his method is closely related to recent interior point methods, in the sense that it also traces the path of centers, even though the tracing is done outside the affine hull of the feasible domain. Also, an equivalence of the fundamental theorems is pointed out.

It is well known (see [2]) that the linear programming problem

$$\text{Minimize} \quad \mathbf{c}^T\mathbf{x} \quad \text{subject to} \quad \mathbf{A}\mathbf{x} \geqslant \mathbf{b}, \quad \mathbf{x} \geqslant \mathbf{0}$$

and its dual

$$\text{Maximize} \quad \mathbf{b}^T\mathbf{y} \quad \text{subject to} \quad \mathbf{A}^T\mathbf{y} \leqslant \mathbf{c}, \quad \mathbf{y} \geqslant \mathbf{0}$$

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can be put together as a linear complementarity problem, i.e., given $\mathbf{M} \in R^{n \times n}$ and $\mathbf{q} \in R^n$, find $\mathbf{z}, \mathbf{w} \in R^n$ such that

$$\mathbf{w} = \mathbf{M}\mathbf{z} + \mathbf{q}, \qquad \mathbf{z}, \mathbf{w} \geqslant 0, \quad \mathbf{z}^T \mathbf{w} = 0,$$

where

$$\mathbf{M} = \begin{bmatrix} \mathbf{O} & -\mathbf{A}^T \\ \mathbf{A} & \mathbf{O} \end{bmatrix}, \qquad \mathbf{q} = \begin{bmatrix} \mathbf{c} \\ -\mathbf{b} \end{bmatrix}, \qquad \mathbf{z} = \begin{bmatrix} \mathbf{x} \\ \mathbf{y} \end{bmatrix}, \qquad \mathbf{w} = \begin{bmatrix} \mathbf{c} - \mathbf{A}^T \mathbf{y} \\ \mathbf{A}\mathbf{x} - \mathbf{b} \end{bmatrix},$$

which can in turn be viewed as a system of piecewise linear equations [3, 8]:

$$\mathbf{\Phi}_M(\mathbf{x}) \equiv \mathbf{x}^+ + \mathbf{M}\mathbf{x}^- = \mathbf{q},$$

where $x_j^+ = \max\{x_j, 0\}$, $x_j^- = \min\{x_j, 0\}$, $\mathbf{x}^{\pm} = (x_1^{\pm}, \dots, x_n^{\pm})^T$, so that $\mathbf{z} = -\mathbf{x}^-$ and $\mathbf{w} = \mathbf{x}^+$. Note that here $\mathbf{x} \in R^n$ is not the same as the $\mathbf{x}$ in the linear programming problem above.

Smale [10] proposed a "regularization" of the piecewise linear system $\mathbf{\Phi}_M(\mathbf{x}) = \mathbf{q}$ as follows. For $a \geqslant 0$, denote

$$\varphi_a^{\pm}(x) = \frac{x \pm \sqrt{x^2 + a^2}}{2}.$$

Approximate $\mathbf{x}^{\pm}$ by

$$\mathbf{\Phi}_a^{\pm}(\mathbf{x}) = \left(\varphi_a^{\pm}(x_1), \dots, \varphi_a^{\pm}(x_n)\right)^T,$$

so $\mathbf{\Phi}_M(\mathbf{x})$ is approximated by

$$\mathbf{\Phi}_a(\mathbf{x}) \equiv \mathbf{\Phi}_a^+(\mathbf{x}) + \mathbf{M}\mathbf{\Phi}_a^-(\mathbf{x}).$$

This approximation is good in the sense that $\mathbf{\Phi}_a(\mathbf{x})$ tends to $\mathbf{\Phi}_M(\mathbf{x})$ uniformly on R^n as $a \to 0$. Incidentally, the other claim that "for each a, $\mathbf{\Phi}_a(\mathbf{x})$ tends to $\mathbf{\Phi}_M(\mathbf{x})$ as $\|\mathbf{x}\| \to \infty$" [10, p. 178, line 8] is incorrect. For example, let

$$\mathbf{M} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Here,

$$\boldsymbol{\Phi}_M(\mathbf{x}) = \left(x_1^+ - x_1^-, x_2^+ + x_2^-\right)^T = \left(|x_1|, x_2\right)^T,$$

whereas

$$\boldsymbol{\Phi}_a(\mathbf{x}) = \left(\varphi_a^+(x_1) - \varphi_a^-(x_1), \varphi_a^+(x_2) + \varphi_a^-(x_2)\right)^T = \left(\sqrt{x_1^2 + a^2}, x_2\right)^T,$$

but x_1 does not necessarily tend to infinity when $\|\mathbf{x}\|$ does.

Smale proposes to solve the linear programming problem as follows. For every sufficiently large $a > \|\mathbf{q}\|$, the zero vector lies in the domain of quadratic convergence of Newton's method for solving the system $\boldsymbol{\Phi}_a(\mathbf{x}) = \mathbf{q} + \boldsymbol{\Phi}_a(\mathbf{0})$. Starting with such an a, a solution for the linear programming problem can be obtained by following the path of solutions of $\boldsymbol{\Phi}_a(\mathbf{x}) = \mathbf{q} + \boldsymbol{\Phi}_a(\mathbf{0})$ as a is driven to zero.

Although it does not seem to be an interior point method, it turns out that Smale's method is very closely related to recent interior path following algorithms.

The "path of centers" for a general linear complementarity problem [7] is defined to be the set of solutions of the following system:

$$\boldsymbol{\eta} = \mathbf{M}\boldsymbol{\xi} + \mathbf{q}, \qquad \boldsymbol{\xi}, \boldsymbol{\eta} > \mathbf{0}, \quad \xi_j \eta_j = \mu \ \ (j = 1, \ldots, n).$$

In the special case of the linear programming problem, this system defines a unique path which is obtained by combining the primal and dual logarithmic barrier trajectories [4, 1, 7, 9, 11]. Given $a > 0$, let us associate with any $\mathbf{x} \in R^n$ a pair of vectors $\boldsymbol{\xi} = \boldsymbol{\xi}(\mathbf{x}) = -\boldsymbol{\Phi}_a^-(\mathbf{x})$ and $\boldsymbol{\eta} = \boldsymbol{\eta}(\mathbf{x}) = \boldsymbol{\Phi}_a^+(\mathbf{x})$. If $\boldsymbol{\Phi}_a(\mathbf{x}) = \mathbf{q}$, then obviously $\boldsymbol{\eta} = \mathbf{M}\boldsymbol{\xi} + \mathbf{q}$ and $\boldsymbol{\xi}, \boldsymbol{\eta} > \mathbf{0}$. Surprisingly, also $\xi_j \eta_j = a^2/4$, and hence the point $(\boldsymbol{\xi}, \boldsymbol{\eta})$ lies on the path of centers where $\mu = a^2/4$.

Conversely, if $(\boldsymbol{\xi}, \boldsymbol{\eta})$ is on the path of centers for a certain value μ, define $\mathbf{x} = \boldsymbol{\eta} - \boldsymbol{\xi}$ and $a = 2\sqrt{\mu}$. We get $\boldsymbol{\Phi}_a^-(\mathbf{x}) = -\boldsymbol{\xi}$, $\boldsymbol{\Phi}_a^+(\mathbf{x}) = \boldsymbol{\eta}$, and $\boldsymbol{\Phi}_a(\mathbf{x}) = \mathbf{q}$. Thus, theorems concerning the path of centers correspond to theorems concerning the set of solutions of the system $\boldsymbol{\Phi}_a(\mathbf{x}) = \mathbf{q}$. The main theorem talks about the existence and uniqueness of the path of centers. This is discussed below.

Interestingly, for *any* $\mathbf{x}$, $\xi_j(\mathbf{x})\eta_j(\mathbf{x}) = \mu$ $(j = 1, \ldots, n)$ and $\boldsymbol{\xi}(\mathbf{x}), \boldsymbol{\eta}(\mathbf{x}) > 0$. However, if $\mathbf{x}$ is not exactly on the path of centers, then $\boldsymbol{\eta}(\mathbf{x}) \neq \mathbf{M}\boldsymbol{\xi}(\mathbf{x}) + \mathbf{q}$. This means that although Smale's method traces the path of centers, it does not do that within the interior of the feasible domain but rather as an exterior point method, although the iterates stay in the positive orthant.

Smale defines

$$\mathscr{U}_M = \{\mathbf{q} : (\exists \mathbf{x}, \mathbf{y} \in R^n)(\mathbf{x}, \mathbf{y} > 0,\ \mathbf{y} = \mathbf{M}\mathbf{x} + \mathbf{q})\},$$

and then proves[1] that if $\mathbf{M}$ is positive semidefinite, then the map $\mathbf{\Phi}_a : R^n \to R^n$ is one-to-one and onto $\mathscr{U}_M$. In fact, Smale proves that $\mathbf{\Phi}_a$ is an analytic diffeomorphism.

An equivalent way of stating that $\mathbf{\Phi}_a$ is one-to-one and onto is as follows. If $\mathbf{M}$ is positive semidefinite and the feasible domain of the LCP has a nonempty interior (i.e., there exist $\mathbf{x}, \mathbf{y} > \mathbf{0}$ such that $\mathbf{y} = \mathbf{M}\mathbf{x} + \mathbf{q}$), then the path of centers exists and is unique, i.e., for every $\mu > 0$, there exists a unique pair $\mathbf{x}, \mathbf{y} > \mathbf{0}$ such that $\mathbf{y} = \mathbf{M}\mathbf{x} + \mathbf{q}$ and $x_j y_j = \mu$ for $j = 1, \ldots, n$.

The equivalent form of the theorem was independently proven by Kojima, Mizuno, and Yoshise [5], continuing the analysis of [7]. The proof uses arguments of convex optimization. The theorem also follows from a more general result on complementarity problems with maximal monotone multifunctions given by McLinden [6]. See Theorems 2 and 3 of [6].

In order to trace the solutions of $\mathbf{\Phi}_a(\mathbf{x}) = \mathbf{q}$, one needs to have as a starting point, an approximate zero[2] of $\mathbf{\Phi}_a(\mathbf{x}) = \mathbf{q}$ for some $a > 0$. Since such a point is not always available, Smale proposes to start with an approximate zero of $\mathbf{\Phi}_a(\mathbf{x}) = \mathbf{q} + \mathbf{\Phi}_a(\mathbf{0})$. In fact, he argues that the zero vector is an approximate zero of the latter for a sufficiently large. The proof of the latter is based on Smale's "α-theorem." Now,

$$\mathbf{\Phi}_a(\mathbf{0}) = \frac{a}{2}(\mathbf{e} - \mathbf{M}\mathbf{e}),$$

so in the space of $(\boldsymbol{\xi}, \boldsymbol{\eta})$, the system $\mathbf{\Phi}_a(\mathbf{x}) = \mathbf{q} + \mathbf{\Phi}_a(\mathbf{0})$ has the form

$$\boldsymbol{\eta} - \mathbf{M}\boldsymbol{\xi} = \mathbf{q} + \sqrt{\mu}\,(\mathbf{e} - \mathbf{M}\mathbf{e}), \qquad \boldsymbol{\xi}, \boldsymbol{\eta} > \mathbf{0}, \quad \xi_j \eta_j = \mu \ \ (j = 1, \ldots, n).$$

Thus, the choice of $\boldsymbol{\xi} = \boldsymbol{\eta} = \sqrt{\mu}\,\mathbf{e}$ gives an approximate solution for a large μ. However, the question of what is a sufficiently large μ can be avoided

[1] At least in one place (p. 189, line 11) the proof relies on the special structure of an LCP derived from a linear programming problem, but it is claimed that it applies to the more general case of a positive semidefinite $\mathbf{M}$.

[2] An approximate zero of differentiable map $\mathbf{F} : R^n \to R^n$ is defined to be a point in the domain of quadratic convergence of Newton's method for the equation $\mathbf{F}(\mathbf{x}) = \mathbf{0}$.

altogether if, instead, we consider the system

$$\boldsymbol{\eta} - \mathbf{M}\boldsymbol{\xi} = \mathbf{q} + \mu(\mathbf{e} - \mathbf{M}\mathbf{e} - \mathbf{q}), \qquad \boldsymbol{\xi}, \boldsymbol{\eta} > \mathbf{0}, \quad \xi_j \eta_j = \mu \ \ (j = 1, \dots, n),$$

where for $\mu = 1$ we have an exact solution available, namely, $\boldsymbol{\xi} = \boldsymbol{\eta} = \mathbf{e}$, and as μ approaches zero, the solution approaches the path of centers. It is not known whether either of these methods can be implemented in polynomial time.

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