A Note on Approximate Linear Programming

Nimrod Megiddo*

October 1991

Abstract. M. Serna recently proved that approximating linear programming is log-space complete for P. This note shows a direct reduction of the exact problem to Serna's approximate one.

Consider the problem: given $\boldsymbol{A} \in Z^{m \times n}$ and $\boldsymbol{b} \in Z^m$, decide whether $D = \{\boldsymbol{x} \in Q^n | \boldsymbol{A}\boldsymbol{x} \le \boldsymbol{b},\ \boldsymbol{x} \ge \boldsymbol{0}\} \ne \emptyset$. If $D \ne \emptyset$, then there exists an $\boldsymbol{x} \in D$ such that $\boldsymbol{x} \le 2^L \boldsymbol{e}$, where L is the length of the input, and $\boldsymbol{e} = (1, \ldots, 1)^T \in Z^n$ (see, *e.g.*, [1]). Consider the problem:

$$(P) \qquad \text{Maximize } t \ \text{ subject to: } \boldsymbol{A}\boldsymbol{x} \le t\,\boldsymbol{b},\ \ \boldsymbol{0} \le \boldsymbol{x} \le t\,2^L \boldsymbol{e},\ \text{ and } 0 \le t \le 1.$$

Since $(\boldsymbol{x}, t) = \boldsymbol{0}$ is feasible in (P), the latter must have an optimal solution $(\boldsymbol{x}_0, t_0)$. It is easy to see that $D \ne \emptyset$ iff $t_0 > 0$, since $D \ne \emptyset$ implies the existence of an $\boldsymbol{x} \in D$ such that $\boldsymbol{x} \le 2^L \boldsymbol{e}$, hence $t_0 = 1$, and if $t_0 > 0$ then $\boldsymbol{x}_0 / t_0 \in D$. Serna's problem [2] calls for finding a t such that $t_0 \ge t \ge \epsilon t_0$, but this means $t > 0$ iff $t_0 > 0$, so this problem is as hard as the exact one. Moreover, for any $\epsilon > 0$ and $p \ge 1$, the other approximate problem of finding $(\boldsymbol{x}, t)$ such that $\|(\boldsymbol{x}_0, t_0)\|_p \ge \|(\boldsymbol{x}, t)\|_p \ge \epsilon \|(\boldsymbol{x}_0, t_0)\|_p$ (see [2]) is also as hard as the exact problem, since

$$\|(\boldsymbol{x}, t)\|_p > 0 \quad \Leftrightarrow \quad \|(\boldsymbol{x}_0, t_0)\|_p > 0 \quad \Leftrightarrow \quad t_0 > 0 \ .$$

References

[1] A. Schrijver, *Theory of linear and integer programming*, Wiley, New York, 1986.

[2] M. Serna, "Approximating linear programming is log-space complete for P," *Information Processing Letters* **37** (1991) 233–236.

*IBM Almaden Research Center, 650 Harry Road, San Jose, California 95120-6099, and School of Mathematical Sciences, Tel Aviv University, Tel Aviv, Israel. Supported in part by ONR Contract N00014-91-C-0026.