1. Introduction

The primal-dual infeasible-interior-point algorithm which we will discuss has stemmed from the primal-dual interior-point algorithm (Megiddo [16], Kojima, Mizuno, and Yoshise [7], Monteiro and Adler [19], Tanabe [22]) for linear programs. It has already been studied by many researchers (Lustig [12], Lustig, Marsten, and Shanno [13], Marsten, Subramanian, Saltzman, Lustig, and Shanno [14], Tanabe [22, 23], Vanderbei and Carpenter [24], etc.), and is known as practically efficient algorithms among numerous variations and extensions of the primal-dual interior-point algorithm.[1] Many numerical studies show that the algorithm solves large scale practical problems very efficiently ([12, 13, 14], etc.). Theoretically, however, neither polynomial-time nor global convergence of the algorithm has been shown. The aim of the present paper is to propose a rule of controlling step lengths to ensure its global convergence.

Let $\boldsymbol{A}$ be an $m \times n$ matrix, $\boldsymbol{b} \in R^m$, and $\boldsymbol{c} \in R^n$. Consider the standard form linear program

$$\begin{array}{lll} \mathcal{P}: & \text{Minimize} & \boldsymbol{c}^T\boldsymbol{x} \\ & \text{subject to} & \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b},\ \boldsymbol{x} \geq \boldsymbol{0}, \end{array}$$

and its dual

$$\begin{array}{lll} \mathcal{D}: & \text{Maximize} & \boldsymbol{b}^T\boldsymbol{y} \\ & \text{subject to} & \boldsymbol{A}^T\boldsymbol{y} + \boldsymbol{z} = \boldsymbol{c},\ \boldsymbol{z} \geq \boldsymbol{0}. \end{array}$$

Throughout the paper, we assume that the matrix $\boldsymbol{A}$ has full row rank, *i.e.*, rank $\boldsymbol{A} = m$. If feasible solutions $\boldsymbol{x}$ of $\mathcal{P}$ and $(\boldsymbol{y}, \boldsymbol{z})$ of $\mathcal{D}$ satisfy $\boldsymbol{x} > \boldsymbol{0}$ and $\boldsymbol{z} > \boldsymbol{0}$, we call them interior feasible solutions of $\mathcal{P}$ and $\mathcal{D}$, respectively. We also say that $(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z})$ is a feasible solution (resp. an interior feasible solution, an optimal solution) of the primal-dual pair of $\mathcal{P}$ and $\mathcal{D}$ if $\boldsymbol{x}$ and $(\boldsymbol{y}, \boldsymbol{z})$ are feasible solutions (resp. interior feasible solutions, optimal solutions) of the problems $\mathcal{P}$ and $\mathcal{D}$, respectively.

Let $(\boldsymbol{x}^1, \boldsymbol{y}^1, \boldsymbol{z}^1)$ be an initial point such that $\boldsymbol{x}^1 > \boldsymbol{0}$ and $\boldsymbol{z}^1 > \boldsymbol{0}$. At each iterate $(\boldsymbol{x}^k, \boldsymbol{y}^k, \boldsymbol{z}^k)$ of a primal-dual infeasible-interior-point algorithm, we compute a Newton direction $(\boldsymbol{\Delta x}, \boldsymbol{\Delta y}, \boldsymbol{\Delta z})$ towards a point on the central trajectory (the path of centers, Megiddo [16], see also Fiacco and McCormick [3]), and then generate a new point $(\boldsymbol{x}^{k+1}, \boldsymbol{y}^{k+1}, \boldsymbol{z}^{k+1})$ such that

$$\begin{aligned} \boldsymbol{x}^{k+1} &= \boldsymbol{x}^k + \alpha_p^k \boldsymbol{\Delta x} > \boldsymbol{0}, \\ \boldsymbol{y}^{k+1} &= \boldsymbol{y}^k + \alpha_d^k \boldsymbol{\Delta y}, \\ \boldsymbol{z}^{k+1} &= \boldsymbol{z}^k + \alpha_d^k \boldsymbol{\Delta z} > \boldsymbol{0}. \end{aligned}$$

Here $\alpha_p^k \geq 0$ and $\alpha_d^k \geq 0$ denote primal and dual step lengths. Thus the infeasible-interior-point algorithm shares this basic structure with many of the primal-dual interior-point

[1] Mehrotra [17] proposed more efficient methods using a predictor-corrector strategy.

algorithms developed so far (Choi, Monma, and Shanno [2], Kojima, Megiddo, Noma, and Yoshise [6], Kojima, Mizuno, and Yoshise [7, 8], Lustig [11, 12], McShane, Monma, and Shanno [15], Mizuno, Todd, and Ye [18], Tanabe [22, 23], Monteiro and Adler [19, 20], Ye [25], Ye, Güler, Tapia, and Zhang [26], etc.).

The generated sequence $\{(\boldsymbol{x}^k, \boldsymbol{y}^k, \boldsymbol{z}^k)\}$ is, however, not restricted to the interior of the feasible region; an iterate $(\boldsymbol{x}^k, \boldsymbol{y}^k, \boldsymbol{z}^k)$ is required to satisfy neither the equality constraints $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$ of $\mathcal{P}$ nor $\boldsymbol{A}^T\boldsymbol{y} + \boldsymbol{z} = \boldsymbol{c}$ of $\mathcal{D}$, but only the positivity $\boldsymbol{x} > \boldsymbol{0}$ and $\boldsymbol{z} > \boldsymbol{0}$. Therefore, we can start from arbitrary $(\boldsymbol{x}^1, \boldsymbol{y}^1, \boldsymbol{z}^1)$ with strictly positive $\boldsymbol{x}^1$ and $\boldsymbol{z}^1$ to approach optimal solutions moving through not only the interior but also the outside of the feasible region of the primal-dual pair of $\mathcal{P}$ and $\mathcal{D}$.

The distinctive feature of the infeasible-interior-point algorithm mentioned above is a main advantage over primal-dual interior-point algorithms. When we apply a primal-dual interior-point algorithm to the problems $\mathcal{P}$ and $\mathcal{D}$, we usually need to prepare an artificial primal-dual pair of linear programs having a known interior feasible solution from which the algorithm starts. Lustig [12] pointed out a drawback of this approach that the artificial primal-dual pair of linear programs involves large constants called the big M and artificial dense columns which often cause numerical instability and computational inefficiency. He derived the limiting feasible direction (Newton direction) as the constant "big M" tends to infinity, and showed that the primal-dual interior-point algorithm using the limiting feasible direction leads to the infeasible-interior-point algorithm. Lustig, Marsten and Shanno [13] showed the equivalence of the the limiting feasible direction and the Newton direction to the Karush-Kuhn-Tucker condition for the linear program $\mathcal{P}$ and $\mathcal{D}$. To mitigate the drawback of interior-point algorithms, Kojima, Mizuno, and Yoshise [9] recently proposed an artificial self-dual linear program with a single big M as well as a numerical method for updating the big M. But the computation of the Newton direction in each iteration of a primal-dual interior-point algorithm applied to the artificial self-dual linear program is still a little more expensive than that of the infeasible-interior-point algorithm. We also mention that the infeasible-interior-point algorithm can be interpreted as an application of an interior-point algorithm to the artificial self-dual linear program (see Section 4 of [9]).

This paper proposes a rule for controlling the step length with which the primal-dual infeasible-interior-point algorithm takes large distinct step lengths α_p^k in the primal space and α_d^k in the dual space, and generates a sequence $\{(\boldsymbol{x}^k, \boldsymbol{y}^k, \boldsymbol{z}^k)\}$ satisfying the following properties:

(a) For any accuracy $\epsilon > 0$ required for the total complementarity, any tolerance $\epsilon_p > 0$ for the primal feasibility, any tolerance ϵ_d for the dual feasibility and any large ω^*, there exists a number k such that after k iterations the algorithm using the rule generates a point $(\boldsymbol{x}^k, \boldsymbol{y}^k, \boldsymbol{z}^k)$ which is either an approximate optimal solution

$(\boldsymbol{x}^k, \boldsymbol{y}^k, \boldsymbol{z}^k)$ satisfying

$$(\boldsymbol{x}^k)^T \boldsymbol{z}^k \leq \epsilon, \ \|\boldsymbol{A}\boldsymbol{x}^k - \boldsymbol{b}\| \leq \epsilon_p \ \text{ and } \|\boldsymbol{A}^T\boldsymbol{y}^k + \boldsymbol{z}^k - \boldsymbol{c}\| \leq \epsilon_d \tag{1}$$

or satisfies

$$\|(\boldsymbol{x}^k, \boldsymbol{z}^k)\|_1 > \omega^* \ . \tag{2}$$

Here $\|\boldsymbol{u}\|_1$ denotes the 1-norm of a vector $\boldsymbol{u} \in R^\ell$, *i.e.*, $\|\boldsymbol{u}\|_1 = \sum_{i=1}^{\ell} \mid u_i \mid$.

(b) If (2) holds, we can derive information on the infeasibility such that the primal-dual pair of $\mathcal{P}$ and $\mathcal{D}$ has no feasible solution in a certain wide region of the primal-dual space. (See [9] for a method of getting such information in interior-point algorithms.)

In Section 2 we give the details of the primal-dual infeasible-interior-point algorithm using the step length control rule, where the inequalities (1) and (2) serve as stopping criteria. In Section 3 we establish that the algorithm enjoys property (a). In Sections 4 and 5, we discuss property (b).

2. An Infeasible-Interior-Point Algorithm

It is convenient to denote the feasible region of the primal-dual pair of $\mathcal{P}$ and $\mathcal{D}$ as $Q_+ \cap S$, where

$$\begin{aligned} Q_+ &= \{(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}) \in R^{n+m+n} : \boldsymbol{x} \geq \boldsymbol{0}, \ \boldsymbol{z} \geq \boldsymbol{0}\}, \\ S &= \{(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}) \in R^{n+m+n} : \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}, \ \boldsymbol{A}^T\boldsymbol{y} + \boldsymbol{z} = \boldsymbol{c}\}. \end{aligned}$$

We also denote by Q_{++} the interior of the set Q_+;

$$Q_{++} = \{(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}) \in R^{n+m+n} : \boldsymbol{x} > \boldsymbol{0}, \ \boldsymbol{z} > \boldsymbol{0}\}.$$

The interior of the feasible region $Q_+ \cap S$ can be denoted by $Q_{++} \cap S$.

Let $0 < \gamma < 1$, $\gamma_p > 0$, and $\gamma_d > 0$. The algorithm generates a sequence $\{(\boldsymbol{x}^k, \boldsymbol{y}^k, \boldsymbol{z}^k)\}$ in the neighborhood

$$\begin{aligned} \mathcal{N} = \{(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}) \in Q_{++} \ : \ & x_i z_i \geq \gamma \boldsymbol{x}^T\boldsymbol{z}/n \ (i = 1, 2, \ldots, n), \\ & \boldsymbol{x}^T\boldsymbol{z} \geq \gamma_p \|\boldsymbol{A}\boldsymbol{x} - \boldsymbol{b}\| \ \text{ or } \ \|\boldsymbol{A}\boldsymbol{x} - \boldsymbol{b}\| \leq \epsilon_p, \\ & \boldsymbol{x}^T\boldsymbol{z} \geq \gamma_d \|\boldsymbol{A}^T\boldsymbol{y} + \boldsymbol{z} - \boldsymbol{c}\| \ \text{ or } \ \|\boldsymbol{A}^T\boldsymbol{y} + \boldsymbol{z} - \boldsymbol{c}\| \leq \epsilon_d\} \end{aligned}$$

of the central trajectory (the path of centers) consisting of the solutions $(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}) \in Q_{++}$ to the system of equations

$$\begin{pmatrix} \boldsymbol{A}\boldsymbol{x} - \boldsymbol{b} \\ \boldsymbol{A}^T\boldsymbol{y} + \boldsymbol{z} - \boldsymbol{c} \\ \boldsymbol{X}\boldsymbol{z} - \mu\boldsymbol{e} \end{pmatrix} = \boldsymbol{0} \tag{3}$$

for all $\mu > 0$. Here $\boldsymbol{X}$ denotes the $n \times n$ diagonal matrix with the coordinates of a vector $\boldsymbol{x} \in R^n$ and $\boldsymbol{e} = (1, \dots, 1)^T \in R^n$.

Let $0 < \beta_1 < \beta_2 < \beta_3 < 1$. At each iteration, we assign the value $\beta_1 (\boldsymbol{x}^k)^T \boldsymbol{z}^k / n$ to the parameter μ, and then compute the Newton direction $(\boldsymbol{\Delta x}, \boldsymbol{\Delta y}, \boldsymbol{\Delta z})$ at $(\boldsymbol{x}^k, \boldsymbol{y}^k, \boldsymbol{z}^k)$ for the system (3) of equations. More precisely, $(\boldsymbol{\Delta x}, \boldsymbol{\Delta y}, \boldsymbol{\Delta z})$ is the unique solution of the system of linear equations

$$\begin{pmatrix} \boldsymbol{A} & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{A}^T & \boldsymbol{I} \\ \boldsymbol{Z}^k & \boldsymbol{0} & \boldsymbol{X}^k \end{pmatrix} \begin{pmatrix} \boldsymbol{\Delta x} \\ \boldsymbol{\Delta y} \\ \boldsymbol{\Delta z} \end{pmatrix} = - \begin{pmatrix} \boldsymbol{A}\boldsymbol{x}^k - \boldsymbol{b} \\ \boldsymbol{A}^T \boldsymbol{y}^k + \boldsymbol{z}^k - \boldsymbol{c} \\ \boldsymbol{X}^k \boldsymbol{z}^k - \mu \boldsymbol{e} \end{pmatrix}. \tag{4}$$

The parameters β_2 and β_3 control the primal and dual step lengths.

We can take an arbitrary initial point $\{(\boldsymbol{x}^1, \boldsymbol{y}^1, \boldsymbol{z}^1)\}$ with $\boldsymbol{x}^1 > \boldsymbol{0}$ and $\boldsymbol{z}^1 > \boldsymbol{0}$, but we must choose the parameters γ, γ_p, γ_d and ω^* such that

$$(\boldsymbol{x}^1, \boldsymbol{y}^1, \boldsymbol{z}^1) \in \mathcal{N} \text{ and } \|(\boldsymbol{x}^1, \boldsymbol{z}^1)\|_1 \leq \omega^*.$$

All the parameters γ, γ_p, γ_d, ϵ, ϵ_p, ϵ_d, ω^*, β_1, β_2 and β_3 may or may not depend on the input data for the problems $\mathcal{P}$ and $\mathcal{D}$.

Now we are ready to state our algorithm.

Algorithm 2.1. Step 1: Let $k = 1$.

Step 2: If (1) or (2) holds then stop.

Step 3: Let $\mu = \beta_1 (\boldsymbol{x}^k)^T \boldsymbol{z}^k / n$. Compute the unique solution $(\boldsymbol{\Delta x}, \boldsymbol{\Delta y}, \boldsymbol{\Delta z})$ at $(\boldsymbol{x}^k, \boldsymbol{y}^k, \boldsymbol{z}^k)$ of the system (4) of equations.

Step 4: Let $\bar{\alpha}^k$ be the maximum of $\tilde{\alpha}$'s ≤ 1 such that the relations

$$\left.\begin{array}{l} (\boldsymbol{x}^k, \boldsymbol{y}^k, \boldsymbol{z}^k) + \alpha(\boldsymbol{\Delta x}, \boldsymbol{\Delta y}, \boldsymbol{\Delta z}) \in \mathcal{N}, \\ (\boldsymbol{x}^k + \alpha \boldsymbol{\Delta x})^T (\boldsymbol{z}^k + \alpha \boldsymbol{\Delta z}) \leq (1 - \alpha(1 - \beta_2))(\boldsymbol{x}^k)^T \boldsymbol{z}^k \end{array}\right\} \tag{5}$$

hold for every $\alpha \in [0, \tilde{\alpha}]$. See Remark 3.1 for the computation of $\bar{\alpha}^k$.

Step 5: Choose a primal step length $\alpha_p^k \in (0, 1]$, a dual step length $\alpha_d^k \in (0, 1]$ and a new iterate $(\boldsymbol{x}^{k+1}, \boldsymbol{y}^{k+1}, \boldsymbol{z}^{k+1})$ such that

$$\left.\begin{array}{l} (\boldsymbol{x}^{k+1}, \boldsymbol{y}^{k+1}, \boldsymbol{z}^{k+1}) = (\boldsymbol{x}^k + \alpha_p^k \boldsymbol{\Delta x}, \boldsymbol{y}^k + \alpha_d^k \boldsymbol{\Delta y}, \boldsymbol{z}^k + \alpha_d^k \boldsymbol{\Delta z}) \in \mathcal{N}, \\ (\boldsymbol{x}^{k+1})^T \boldsymbol{z}^{k+1} \leq (1 - \bar{\alpha}^k (1 - \beta_3))(\boldsymbol{x}^k)^T \boldsymbol{z}^k. \end{array}\right\} \tag{6}$$

Step 6: Increase k by 1. Go to Step 2.

Since $0 < \beta_2 < \beta_3 < 1$, the common value $\bar{\alpha}^k$ is always available for both the primal step length α_p^k and the dual step length α_d^k although we can assign distinct values to them. In the next section, we will show existence of a positive number α^* such that $\bar{\alpha}^k$ is not less than α^* for every k as long as the iteration continues. This will lead to a finite termination of the algorithm at Step 2.

We consider how large step lengths α_p^k and α_d^k we can choose subject to the condition (6) at Step 5 of the algorithm. The second inequality of (6) requires a reduction $(1-\bar{\alpha}^k(1-\beta_3))$ in the total complementarity $\boldsymbol{x}^T\boldsymbol{z}$. By the definition of $\bar{\alpha}^k$ and $0 < \beta_2 < \beta_3 < 1$, a bigger reduction $(1-\bar{\alpha}^k(1-\beta_2))$ than $(1-\bar{\alpha}^k(1-\beta_3))$ is always possible. So the inequality seems reasonable. If we take a positive β_3 less than but sufficiently close to 1, the inequality does little harm to take large step lengths.

Now we focus our attention on the first constraint of (6) which the step lengths α_p^k and α_d^k must satisfy. The definition of $\mathcal{N}$ consists of three kinds of relations

$$x_i z_i \geq \gamma \boldsymbol{x}^T\boldsymbol{z}/n \ (i = 1, 2, \ldots, n), \tag{7}$$
$$\boldsymbol{x}^T\boldsymbol{z} \geq \gamma_p \|\boldsymbol{A}\boldsymbol{x} - \boldsymbol{b}\| \ \text{ or } \ \|\boldsymbol{A}\boldsymbol{x} - \boldsymbol{b}\| \leq \epsilon_p, \tag{8}$$
$$\boldsymbol{x}^T\boldsymbol{z} \geq \gamma_d \|\boldsymbol{A}^T\boldsymbol{y} + \boldsymbol{z} - \boldsymbol{c}\| \ \text{ or } \ \|\boldsymbol{A}^T\boldsymbol{y} + \boldsymbol{z} - \boldsymbol{c}\| \leq \epsilon_d. \tag{9}$$

As in the primal-dual interior-point algorithms (Kojima, Mizuno, and Yoshise [7], Mizuno, Todd, and Ye [18], etc.), the inequalities (7) prevent the generated sequence $\{(\boldsymbol{x}^k, \boldsymbol{y}^k, \boldsymbol{z}^k)\}$ from reaching the boundary of Q_{++} before the total complementarity $(\boldsymbol{x}^k)^T\boldsymbol{z}^k$ attains 0. The other relations (8) and (9) play the role of excluding the possibility that the generated sequence $\{(\boldsymbol{x}^k, \boldsymbol{y}^k, \boldsymbol{z}^k)\}$ might converge to an infeasible complementary solution $(\boldsymbol{x}^*, \boldsymbol{y}^*, \boldsymbol{z}^*) \in Q_+, \ (\boldsymbol{x}^*)^T\boldsymbol{z}^* = 0$ such that

$$\|\boldsymbol{A}\boldsymbol{x}^* - \boldsymbol{b}\| \geq \epsilon_p \ \text{ and/or } \ \|\boldsymbol{A}^T\boldsymbol{y}^* + \boldsymbol{z}^* - \boldsymbol{c}\| \geq \epsilon_d.$$

Besides the primal feasibility tolerance $\epsilon_p > 0$ and the dual feasibility tolerance $\epsilon_d > 0$, which we assume fixed in what follows, the set $\mathcal{N}$ involves the positive parameters γ, γ_p, and γ_d. To clarify the dependency on these parameters, we will write the set as $\mathcal{N}(\gamma, \gamma_p, \gamma_d)$. Then

$$\mathcal{N}(\gamma, \gamma_p, \gamma_d) \supset \mathcal{N}(\gamma', \gamma_p', \gamma_d') \ \text{ if } \ \boldsymbol{0} < (\gamma, \gamma_p, \gamma_d) \leq (\gamma', \gamma_p', \gamma_d'),$$
$$\cup\{\mathcal{N}(\gamma, \gamma_p, \gamma_d) \ : \ (\gamma, \gamma_p, \gamma_d) > \boldsymbol{0}\} \ = \ Q_{++}.$$

Therefore, as we take smaller positive γ, γ_p and γ_d, the set $\mathcal{N}(\gamma, \gamma_p, \gamma_d)$ covers a larger subregion of Q_{++}; hence, we can take larger step lengths α_p and α_d satisfying (6).

McShane, Monma, and Shanno [15], proposed taking large step lengths α_p^k and α_d^k such that

$$\alpha_p^k = 0.9995\hat{\alpha}_p^k \ \text{ and } \alpha_d^k = 0.9995\hat{\alpha}_d^k, \tag{10}$$

where

$$\hat{\alpha}_p^k = \max\{\alpha : \boldsymbol{x}^k + \alpha\boldsymbol{\Delta x} \geq \boldsymbol{0}\},$$
$$\hat{\alpha}_d^k = \max\{\alpha : \boldsymbol{z}^k + \alpha\boldsymbol{\Delta z} \geq \boldsymbol{0}\}.$$

This choice of the step lengths is known to work very efficiently in practice ([12, 13, 14, 15], etc.), but has not been shown to ensure the global convergence. The above observation on the set $\mathcal{N}(\gamma, \gamma_p, \gamma_d)$ suggests a combination of their step lengths with ours to ensure the global convergence: Take the large step lengths α_p^k and α_d^k given in (10) when they satisfy (6), and the common step length $\alpha_p^k = \alpha_d^k = \bar{\alpha}^k$ otherwise. If we choose sufficiently small positive γ, γ_p and γ_d, we can expect that the large step lengths α_p^k and α_d^k given in (10) usually satisfy (6).

Remark 2.2. Kojima, Megiddo, and Noma [5] proposed a continuation method that traces a trajectory leading to a solution of the complementarity problem. If we take a positive γ less than but close to 1 and small positive γ_p, γ_d, then the set $\mathcal{N}$ constitutes a narrow neighborhood of the central trajectory. In this case our infeasible-interior-point algorithm may be regarded as path-following or continuation method, which generates a sequence $\{(\boldsymbol{x}^k, \boldsymbol{y}^k, \boldsymbol{z}^k)\}$ tracing the central trajectory in its narrow neighborhood. It makes a main difference between the Kojima-Megiddo-Noma continuation algorithm and our algorithm that the trajectory traced by their algorithm runs through the outside of the feasible region while the central trajectory traced by our algorithm runs through the interior of the feasible region. See also Kojima, Megiddo, and Mizuno [4] for a more general framework of continuation methods for complementarity problems.

Remark 2.3. If we restricted $\mathcal{N}$ to the set S, $\mathcal{N}$ would turn out to be

$$\begin{aligned}\{(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}) \in Q_{++} \cap S \;\; : \;\; & \boldsymbol{x} \geq 0,\; \boldsymbol{z} \geq 0, \\ & x_i z_i \geq \gamma \boldsymbol{x}^T \boldsymbol{z}/n \;\; (i = 1, 2, \ldots, n)\}.\end{aligned}$$

This type of neighborhood of the central trajectory relative to S has been utilized in many primal-dual interior-point algorithms ([6, 7, 8, 18, 19, 20, 22, 23], etc.). Specifically, it coincides with the one used by Mizuno, Todd, and Ye [18].

The generated sequence $\{(\boldsymbol{x}^k, \boldsymbol{y}^k, \boldsymbol{z}^k)\}$ satisfies the following relations which will be used in the succeeding sections:

$$\left.\begin{aligned} &\boldsymbol{A}(\boldsymbol{x}^k + \alpha \boldsymbol{\Delta x}) - \boldsymbol{b} = (1-\alpha)(\boldsymbol{A}\boldsymbol{x}^k - \boldsymbol{b}) \\ &\boldsymbol{A}^T(\boldsymbol{y}^k + \alpha \boldsymbol{\Delta y}) + (\boldsymbol{z}^k + \alpha \boldsymbol{\Delta z}) - \boldsymbol{c} \\ &\qquad\qquad = (1-\alpha)(\boldsymbol{A}^T\boldsymbol{y}^k + \boldsymbol{z}^k - \boldsymbol{c}) \end{aligned}\right\} \text{ for every } \alpha \geq 0, \tag{11}$$

$$(\boldsymbol{x}^{k+1})^T \boldsymbol{z}^{k+1} \leq (1 - \bar{\alpha}^k(1-\beta_3))(\boldsymbol{x}^k)^T \boldsymbol{z}^k \leq (\boldsymbol{x}^1)^T \boldsymbol{z}^1, \tag{12}$$

$$\left.\begin{aligned} &x_i^k z_i^k \geq \gamma (\boldsymbol{x}^k)^T \boldsymbol{z}^k/n \;\; (i = 1, 2, \ldots, n), \\ &(\boldsymbol{x}^k)^T \boldsymbol{z}^k \geq \gamma_p \|\boldsymbol{A}\boldsymbol{x}^k - \boldsymbol{b}\| \;\text{ or }\; \|\boldsymbol{A}\boldsymbol{x}^k - \boldsymbol{b}\| \leq \epsilon_p, \\ &(\boldsymbol{x}^k)^T \boldsymbol{z}^k \geq \gamma_d \|\boldsymbol{A}^T\boldsymbol{y}^k + \boldsymbol{z}^k - \boldsymbol{c}\| \;\text{ or }\; \|\boldsymbol{A}^T\boldsymbol{y}^k + \boldsymbol{z}^k - \boldsymbol{c}\| \leq \epsilon_d, \end{aligned}\right\} \tag{13}$$

$$\left.\begin{aligned} &(\boldsymbol{x}^k)^T \Delta \boldsymbol{z} + \Delta \boldsymbol{x}^T \boldsymbol{z}^k = -(1-\beta_1)(\boldsymbol{x}^k)^T \boldsymbol{z}^k, \\ &x_i^k \Delta z_i + \Delta x_i z_i^k = \beta_1 (\boldsymbol{x}^k)^T \boldsymbol{z}^k/n - x_i^k z_i^k. \end{aligned}\right\} \tag{14}$$

Here the equalities in (11) follow from the Newton equation (4), the inequality (12) follows from Step 5 of the algorithm (see the last inequality of (6)), the inequalities in (13) follow from $(\boldsymbol{x}^k, \boldsymbol{y}^k, \boldsymbol{z}^k) \in \mathcal{N}$, and the equalities in (14) follow from the Newton equation (4) with $\mu = \beta_1 (\boldsymbol{x}^k)^T \boldsymbol{z}^k$.

3. Global Convergence

In this section we show that the algorithm presented in the previous section terminates at Step 2 in a finite number of iterations for any positive ϵ, ϵ_p, ϵ_d and ω^* associated with its stopping criteria (1) and (2). We assume, on the contrary, that the algorithm never stops and derive a contradiction.

We first observe that in addition to (11) $\sim$ (14) the inequalities

$$(\boldsymbol{x}^k)^T \boldsymbol{z}^k \geq \epsilon^* \quad \text{and} \quad \|(\boldsymbol{x}^k, \boldsymbol{z}^k)\|_1 \leq \omega^* \tag{15}$$

hold for every k $(k = 1, 2, \ldots)$, where

$$\epsilon^* = \min\{\epsilon,\ \gamma_p \epsilon_p,\ \gamma_d \epsilon_d\}, \tag{16}$$

because otherwise $(\boldsymbol{x}^k, \boldsymbol{y}^k, \boldsymbol{z}^k)$ would satisfy either of the stopping criteria (1) and (2) for some k. Hence, the entire sequence $\{(\boldsymbol{x}^k, \boldsymbol{y}^k, \boldsymbol{z}^k)\}$ lies in the compact set

$$\mathcal{N}^* = \{(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}) \in \mathcal{N} : \boldsymbol{x}^T \boldsymbol{z} \geq \epsilon^* \ \text{ and } \ \|(\boldsymbol{x}, \boldsymbol{z})\|_1 \leq \omega^*\}.$$

On the other hand, the Newton direction $(\boldsymbol{\Delta x}, \boldsymbol{\Delta y}, \boldsymbol{\Delta z})$ determined by the system (4) of equations is a continuous function of the location of $(\boldsymbol{x}^k, \boldsymbol{y}^k, \boldsymbol{z}^k) \in \mathcal{N}^*$. This is easily seen because the coefficient matrix on the left hand side of (4) is nonsingular for any $(\boldsymbol{x}^k, \boldsymbol{y}^k, \boldsymbol{z}^k) \in \mathcal{N}^*$, and the coefficient matrix as well as the right hand side of the system (4) is continuous in $(\boldsymbol{x}^k, \boldsymbol{y}^k, \boldsymbol{z}^k) \in \mathcal{N}^*$. Therefore the Newton direction $(\boldsymbol{\Delta x}, \boldsymbol{\Delta y}, \boldsymbol{\Delta z})$ is uniformly bounded for all $(\boldsymbol{x}^k, \boldsymbol{y}^k, \boldsymbol{z}^k)$ over the compact set $\mathcal{N}^*$. Therefore we can find a positive constant η such that the Newton direction $(\boldsymbol{\Delta x}, \boldsymbol{\Delta y}, \boldsymbol{\Delta z})$ computed at Step 3 of every iteration satisfies the inequalities

$$| \ \Delta x_i \Delta z_i - \gamma \boldsymbol{\Delta x}^T \boldsymbol{\Delta z}/n \ | \leq \eta \quad \text{and} \quad | \ \boldsymbol{\Delta x}^T \boldsymbol{\Delta z} \ | \leq \eta, \tag{17}$$

which will be utilized below.

Let k be fixed arbitrarily. Define the real-valued quadratic functions f_i $(i = 1, 2, \ldots, n)$, g_p, g_d, and h as follows.

$$\begin{aligned}
f_i(\alpha) &= (x_i^k + \alpha \Delta x_i)(z_i^k + \alpha \Delta z_i) - \gamma (\boldsymbol{x}^k + \alpha \boldsymbol{\Delta x})^T (\boldsymbol{z}^k + \alpha \boldsymbol{\Delta z})/n, \\
g_p(\alpha) &= (\boldsymbol{x}^k + \alpha \boldsymbol{\Delta x})^T (\boldsymbol{z}^k + \alpha \boldsymbol{\Delta z}) - \gamma_p (1 - \alpha) \|\boldsymbol{A}\boldsymbol{x}^k - \boldsymbol{b}\|, \\
g_d(\alpha) &= (\boldsymbol{x}^k + \alpha \boldsymbol{\Delta x})^T (\boldsymbol{z}^k + \alpha \boldsymbol{\Delta z}) - \gamma_d (1 - \alpha) \|\boldsymbol{A}^T \boldsymbol{y}^k - \boldsymbol{z}^k - \boldsymbol{c}\|, \\
h(\alpha) &= (1 - \alpha(1 - \beta_2))(\boldsymbol{x}^k)^T \boldsymbol{z}^k - (\boldsymbol{x}^k + \alpha \boldsymbol{\Delta x})^T (\boldsymbol{z}^k + \alpha \boldsymbol{\Delta z}).
\end{aligned}$$

By (11), we see that the terms

$$(1-\alpha)\|\boldsymbol{A}\boldsymbol{x}^k - \boldsymbol{b}\| \ \text{ and } (1-\alpha)\|\boldsymbol{A}^T\boldsymbol{y}^k - \boldsymbol{z}^k - \boldsymbol{c}\|$$

coincide with

$$\|\boldsymbol{A}(\boldsymbol{x}^k + \alpha\boldsymbol{\Delta x}) - \boldsymbol{b}\| \ \text{ and } \|\boldsymbol{A}^T(\boldsymbol{y}^k + \alpha\boldsymbol{\Delta y}) + (\boldsymbol{z}^k + \alpha\boldsymbol{\Delta z}) - \boldsymbol{c}\|,$$

respectively. Hence, we can rewrite the relation (5) to determine the $\bar{\alpha}^k$ as

$$\begin{array}{l}
f_i(\alpha) \geq 0 \ \ (i = 1, 2, \ldots, n), \\
g_p(\alpha) \geq 0 \ \text{ or } \ (1-\alpha)\|\boldsymbol{A}\boldsymbol{x}^k - \boldsymbol{b}\| \leq \epsilon_p, \\
g_d(\alpha) \geq 0 \ \text{ or } \ (1-\alpha)\|\boldsymbol{A}^T\boldsymbol{y}^k - \boldsymbol{z}^k - \boldsymbol{c}\| \leq \epsilon_d, \\
h(\alpha) \geq 0.
\end{array}$$

Remark 3.1. Since all the functions in the inequalities above are linear or quadratic, we can easily compute the value of $\bar{\alpha}^k$ by solving them for α.

We can verify that for every i $(i = 1, 2, \ldots, n)$ and $\alpha \in [0, 1]$,

$$\begin{array}{rcl}
f_i(\alpha) & = & x_i^k z_i^k + \alpha(z_i^k \Delta x_i + x_i^k \Delta z_i) + \alpha^2 \Delta x_i \Delta z_i \\
& & -\gamma\left((\boldsymbol{x}^k)^T\boldsymbol{z}^k + \alpha((\boldsymbol{z}^k)^T\boldsymbol{\Delta x} + (\boldsymbol{x}^k)^T\boldsymbol{\Delta z}) + \alpha^2(\boldsymbol{\Delta x})^T\boldsymbol{\Delta z}\right)/n \\
& = & x_i^k z_i^k - \alpha(x_i^k z_i^k - \beta_1(\boldsymbol{x}^k)^T\boldsymbol{z}^k/n) + \alpha^2 \Delta x_i \Delta z_i \\
& & -\gamma\left((\boldsymbol{x}^k)^T\boldsymbol{z}^k - \alpha(1-\beta_1)(\boldsymbol{x}^k)^T\boldsymbol{z}^k) + \alpha^2(\boldsymbol{\Delta x})^T\boldsymbol{\Delta z}\right)/n \quad \text{(by (14))} \\
& = & x_i^k z_i^k(1-\alpha) + \alpha\beta_1(\boldsymbol{x}^k)^T\boldsymbol{z}^k/n + \alpha^2 \Delta x_i \Delta z_i \\
& & -\gamma\left((\boldsymbol{x}^k)^T\boldsymbol{z}^k(1-\alpha) + \alpha\beta_1(\boldsymbol{x}^k)^T\boldsymbol{z}^k) + \alpha^2(\boldsymbol{\Delta x})^T\boldsymbol{\Delta z}\right)/n \\
& = & (x_i^k z_i^k - \gamma(\boldsymbol{x}^k)^T\boldsymbol{z}^k/n)(1-\alpha) + \beta_1(1-\gamma)((\boldsymbol{x}^k)^T\boldsymbol{z}^k/n))\alpha \\
& & +(\Delta x_i \Delta z_i - \gamma\boldsymbol{\Delta x}^T\boldsymbol{\Delta z}/n)\alpha^2 \\
& \geq & \beta_1(1-\gamma)(\epsilon^*/n)\alpha - \eta\alpha^2 \quad \text{(by (13), (15) and (17))}
\end{array}$$

Similarly, for every $\alpha \in [0, 1]$,

$$\begin{array}{ll}
g_p(\alpha) \geq \beta_1\epsilon^*\alpha - \eta\alpha^2 & \text{if } g_p(0) = (\boldsymbol{x}^k)^T\boldsymbol{z}^k - \gamma_p\|\boldsymbol{A}\boldsymbol{x}^k - \boldsymbol{b}\| \geq 0, \\
(1-\alpha)\|\boldsymbol{A}\boldsymbol{x}^k - \boldsymbol{b}\| \leq \epsilon_p & \text{if } g_p(0) < 0, \\
g_d(\alpha) \geq \beta_1\epsilon^*\alpha - \eta\alpha^2 & \text{if } g_d(0) = (\boldsymbol{x}^k)^T\boldsymbol{z}^k - \gamma_d\|\boldsymbol{A}^T\boldsymbol{y}^k + \boldsymbol{z}^k - \boldsymbol{c}\| \geq 0, \\
(1-\alpha)\|\boldsymbol{A}^T\boldsymbol{y}^k + \boldsymbol{z}^k - \boldsymbol{c}\| \leq \epsilon_d \ \ \text{if } g_d(0) < 0, & \\
h(\alpha) \geq (\beta_2 - \beta_1)\epsilon^*\alpha - \eta\alpha^2. &
\end{array}$$

Hence, letting

$$\alpha^* = \min\left\{1, \ \frac{\beta_1(1-\gamma)\epsilon^*}{n\eta}, \ \frac{\beta_1\epsilon^*}{\eta}, \ \frac{(\beta_2 - \beta_1)\epsilon^*}{\eta}\right\},$$

we obtain that the inequalities

$$\begin{array}{lll}
f_i(\alpha) \geq 0 \ \ (i = 1, 2, \ldots, n), & & \\
g_p(\alpha) \geq 0 & \text{if} & g_p(0) \geq 0, \\
(1-\alpha)\|\boldsymbol{A}\boldsymbol{x}^k - \boldsymbol{b}\| \leq \epsilon_p & \text{if} & g_p(0) < 0, \\
g_d(\alpha) \geq 0 & \text{if} & g_d(0) \geq 0, \\
(1-\alpha)\|\boldsymbol{A}^T\boldsymbol{y}^k + \boldsymbol{z}^k - \boldsymbol{c}\| \leq \epsilon_d & \text{if} & g_d(0) < 0, \\
h(\alpha) \geq 0 & &
\end{array}$$

hold for every $\alpha \in [0, \alpha^*]$. By the construction of the real-valued functions f_i $(i = 1, 2, \ldots, n)$, g_p, g_d, and h, this can be restated as: the relation (5) holds for every $\alpha \in [0, \alpha^*]$. Thus we have shown that the inequality $\bar{\alpha}^k \geq \alpha^*$ holds for every k $(k = 1, 2, \cdots)$.

Finally, by the inequality (12) and $\bar{\alpha}^k \geq \alpha^*$,

$$(\boldsymbol{x}^k)^T\boldsymbol{z}^k \ \ \leq \ \ (1 - (\alpha^*)(1-\beta_3))^{k-1}(\boldsymbol{x}^1)^T\boldsymbol{z}^1 \ (k = 2, 3, \ldots).$$

Obviously, the right-hand side of the inequality converges to zero as k tends to ∞; hence, so does the left-hand side. This contradicts the first inequality of (15).

4. Detecting Infeasibility – I

We showed in the previous section that the algorithm stops at Step 2 in a finite number of iterations for any small $\epsilon > 0$, $\epsilon_p > 0$, $\epsilon_d > 0$, and any large $\omega^* > 0$. If the algorithm stops with the stopping criterion (1), we obtain an approximate optimal solution $(\boldsymbol{x}^k, \boldsymbol{y}^k, \boldsymbol{z}^k)$ of the primal-dual pair of $\mathcal{P}$ and $\mathcal{D}$. In this section and the next one we will derive information on the infeasibility of the primal-dual pair of $\mathcal{P}$ and $\mathcal{D}$ when the algorithm stops with the criterion (2).

For every pair of nonnegative real numbers δ and ω, define

$$S(\delta, \omega) = \{(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}) \in Q_+ : \delta\boldsymbol{e} \leq \boldsymbol{x}, \ \delta\boldsymbol{e} \leq \boldsymbol{z} \ \text{ and } \|(\boldsymbol{x}, \boldsymbol{z})\|_1 \leq \omega\}.$$

We will be concerned with the question whether the region $S(\delta, \omega)$ contains a feasible solution $(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z})$ of the primal-dual pair of $\mathcal{P}$ and $\mathcal{D}$.

Theorem 4.1. *Take positive numbers δ, ω, and ω^* satisfying*

$$(\boldsymbol{x}^1, \boldsymbol{y}^1, \boldsymbol{z}^1) \in S(\delta, \omega) \quad \textit{and} \quad \frac{(\omega)^2 + (\boldsymbol{x}^1)^T\boldsymbol{z}^1}{\delta} \leq \omega^*.$$

Assume that the algorithm has stopped at Step 2 with the stopping criterion (2), i.e. $\|(\boldsymbol{x}^k, \boldsymbol{z}^k)\|_1 > \omega^*$. Then, the region $S(\delta, \omega)$ contains no feasible solution $(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z})$ of the primal-dual pair of $\mathcal{P}$ and $\mathcal{D}$.

In the remainder of this section we prove Theorem 4.1. It is convenient to introduce the following primal-dual pair of parametric linear programs with parameters $\theta_p,\ \theta_d \in [0,1]$:

$$\begin{array}{lll} \mathcal{P}(\theta_p,\theta_d) & \text{Minimize} & \boldsymbol{c}(\theta_d)^T\boldsymbol{x} \\ & \text{subject to} & \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}(\theta_p),\ \boldsymbol{x} \geq \boldsymbol{0}. \\ \mathcal{D}(\theta_p,\theta_d) & \text{Maximize} & \boldsymbol{b}(\theta_p)^T\boldsymbol{y} \\ & \text{subject to} & \boldsymbol{A}^T\boldsymbol{y} + \boldsymbol{z} = \boldsymbol{c}(\theta_d),\ \boldsymbol{z} \geq \boldsymbol{0}. \end{array}$$

Here,

$$\begin{aligned} \boldsymbol{b}(\theta) &= \theta\boldsymbol{A}\boldsymbol{x}^1 + (1-\theta)\boldsymbol{b}, \\ \boldsymbol{c}(\theta) &= \theta(\boldsymbol{A}^T\boldsymbol{y}^1 + \boldsymbol{z}^1) + (1-\theta)\boldsymbol{c}. \end{aligned}$$

We obtain from (11) that

$$\boldsymbol{A}\boldsymbol{x}^k - \boldsymbol{b} \ = \ \prod_{j=1}^{k-1}(1-\alpha_p^j)(\boldsymbol{A}\boldsymbol{x}^1 - \boldsymbol{b}),$$

and that

$$\boldsymbol{A}^T\boldsymbol{y}^k + \boldsymbol{z}^k - \boldsymbol{c} \ = \ \prod_{j=1}^{k-1}(1-\alpha_d^j)(\boldsymbol{A}^T\boldsymbol{y}^1 + \boldsymbol{z}^1 - \boldsymbol{c}).$$

These properties were shown in the paper [1]. Geometrically, this implies that the point $\boldsymbol{A}\boldsymbol{x}^k - \boldsymbol{b}$ lies in the line segment connecting the point $\boldsymbol{A}\boldsymbol{x}^1 - \boldsymbol{b}$ and the origin $\boldsymbol{0} \in \boldsymbol{R}^m$, and that the point $\boldsymbol{A}^T\boldsymbol{y}^k + \boldsymbol{z}^k - \boldsymbol{c}$ lies in the line segment connecting the point $\boldsymbol{A}^T\boldsymbol{y}^1 + \boldsymbol{z}^1 - \boldsymbol{c}$ and the origin $\boldsymbol{0} \in \boldsymbol{R}^n$. Specifically, we know that

$$\begin{array}{l} \boldsymbol{A}\boldsymbol{x}^k - \boldsymbol{b} = 0 \ \text{ for every } k \geq 1 \text{ if } \ \boldsymbol{A}\boldsymbol{x}^1 - \boldsymbol{b} = \boldsymbol{0}, \\ \boldsymbol{A}^T\boldsymbol{y}^k + \boldsymbol{z}^k - \boldsymbol{c} = 0 \ \text{ for every } k \geq 1 \text{ if } \ \boldsymbol{A}^T\boldsymbol{y}^1 + \boldsymbol{z}^k - \boldsymbol{c} = \boldsymbol{0}. \end{array}$$

Hence, by defining

$$\begin{aligned} \theta_p^k &= \begin{cases} \|\boldsymbol{A}\boldsymbol{x}^k - \boldsymbol{b}\|/\|\boldsymbol{A}\boldsymbol{x}^1 - \boldsymbol{b}\| & \text{if } \|\boldsymbol{A}\boldsymbol{x}^1 - \boldsymbol{b}\| > 0, \\ 0 & \text{if } \|\boldsymbol{A}\boldsymbol{x}^1 - \boldsymbol{b}\| = 0, \end{cases}, \\ \theta_d^k &= \begin{cases} \|\boldsymbol{A}^T\boldsymbol{y}^k + \boldsymbol{z}^k - \boldsymbol{c}\|/\|\boldsymbol{A}^T\boldsymbol{y}^1 + \boldsymbol{z}^1 - \boldsymbol{c}\| & \text{if } \|\boldsymbol{A}^T\boldsymbol{y}^1 + \boldsymbol{z}^1 - \boldsymbol{c}\| > 0, \\ 0 & \text{if } \|\boldsymbol{A}^T\boldsymbol{y}^1 + \boldsymbol{z}^1 - \boldsymbol{c}\| = 0, \end{cases} \end{aligned}$$

we have

$$\left.\begin{array}{l} \boldsymbol{A}\boldsymbol{x}^k - \boldsymbol{b} = \theta_p^k(\boldsymbol{A}\boldsymbol{x}^1 - \boldsymbol{b}), \\ \boldsymbol{A}^T\boldsymbol{y}^k + \boldsymbol{z}^k - \boldsymbol{c} = \theta_d^k(\boldsymbol{A}^T\boldsymbol{y}^1 + \boldsymbol{z}^1 - \boldsymbol{c}) \end{array}\right\} \tag{18}$$

for each k. Thus, we can measure the primal and dual infeasibilities of the k'th iterate $(\boldsymbol{x}^k, \boldsymbol{y}^k, \boldsymbol{z}^k)$ in terms of θ_p^k and θ_d^k, respectively. It should be noted that both of the sequences $\{\theta_p^k\}$ and $\{\theta_d^k\}$ are monotone nonincreasing. The relation (18) together with $\boldsymbol{x}^k > \boldsymbol{0}$ and $\boldsymbol{z}^k > \boldsymbol{0}$ implies that $(\boldsymbol{x}^k, \boldsymbol{y}^k, \boldsymbol{z}^k)$ is an interior feasible solution of the primal-dual pair of $\mathcal{P}(\theta_p^k, \theta_d^k)$ and $\mathcal{D}(\theta_p^k, \theta_d^k)$. We also observe that if $\alpha_p^k = 1$ (or $\alpha_d^k = 1$) holds at a k'th iteration then, for every j ($j = k+1, k+2, \ldots$), $\boldsymbol{x}^j$ is a feasible solution of $\mathcal{P}$ and $\theta_p^j = 0$ ($(\boldsymbol{y}^j, \boldsymbol{z}^j)$ is a feasible solution of $\mathcal{D}$ and $\theta_d^j = 0$).

The following lemma plays an essential role in proving not only Theorem 4.1 but also the theorems in the next section.

Lemma 4.2. *Suppose that $\|(\boldsymbol{x}^k, \boldsymbol{z}^k)\|_1 > \omega^*$. Assume that $(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z})$ is a feasible solution of the primal-dual pair of $\mathcal{P}(\theta_p^k, \theta_d^k)$ and $\mathcal{D}(\theta_p^k, \theta_d^k)$ satisfying*

$$\|\boldsymbol{x}\|_1 \leq \omega_p, \ \|\boldsymbol{z}\|_1 \leq \omega_d, \ \delta \boldsymbol{e} \leq \boldsymbol{x}, \ \delta \boldsymbol{e} \leq \boldsymbol{z}. \tag{19}$$

Then

$$\omega_p \omega_d + (\boldsymbol{x}^1)^T \boldsymbol{z}^1 > \delta \omega^*.$$

Proof: By (18) and the assumption of the lemma, we see that both $(\boldsymbol{x}^k, \boldsymbol{y}^k, \boldsymbol{z}^k)$ and $(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z})$ are feasible solutions of the primal-dual pair of $\mathcal{P}(\theta_p^k, \theta_d^k)$ and $\mathcal{D}(\theta_p^k, \theta_d^k)$. It follows that

$$\boldsymbol{A}(\boldsymbol{x} - \boldsymbol{x}^k) = \boldsymbol{0} \ \text{ and } \ \boldsymbol{z} - \boldsymbol{z}^k = -\boldsymbol{A}^T(\boldsymbol{y} - \boldsymbol{y}^k).$$

Hence,

$$(\boldsymbol{x} - \boldsymbol{x}^k)^T (\boldsymbol{z} - \boldsymbol{z}^k) = 0,$$

or

$$\boldsymbol{x}^T \boldsymbol{z} + (\boldsymbol{x}^k)^T \boldsymbol{z}^k = \boldsymbol{x}^T \boldsymbol{z}^k + (\boldsymbol{x}^k)^T \boldsymbol{z}.$$

Thus we obtain

$$\begin{array}{rcl}
\omega_p \omega_d + (\boldsymbol{x}^1)^T \boldsymbol{z}^1 & \geq & \boldsymbol{x}^T \boldsymbol{z} + (\boldsymbol{x}^k)^T \boldsymbol{z}^k \\
& & (\text{since } \omega_p \geq \|\boldsymbol{x}\|_1, \, \omega_d \geq \|\boldsymbol{z}\|_1 \text{ and } (\boldsymbol{x}^1)^T \boldsymbol{z}^1 \geq (\boldsymbol{x}^k)^T \boldsymbol{z}^k) \\
& = & \boldsymbol{x}^T \boldsymbol{z}^k + (\boldsymbol{x}^k)^T \boldsymbol{z} \\
& \geq & \delta \boldsymbol{e}^T \boldsymbol{z}^k + \delta \boldsymbol{e}^T \boldsymbol{x}^k \ \ (\text{since } \boldsymbol{x} \geq \delta \boldsymbol{e} \text{ and } \boldsymbol{z} \geq \delta \boldsymbol{e}) \\
& = & \delta \|(\boldsymbol{x}^k, \boldsymbol{z}^k)\|_1 > \delta \omega^*
\end{array}$$

∎

Proof of Theorem 4.1: Assume, on the contrary, that the region $S(\delta, \omega)$ contains a feasible solution $(\tilde{\boldsymbol{x}}, \tilde{\boldsymbol{y}}, \tilde{\boldsymbol{z}})$ of the primal-dual pair of $\mathcal{P}$ and $\mathcal{D}$. Let

$$\boldsymbol{x} = (1 - \theta_p^k)\tilde{\boldsymbol{x}} + \theta_p^k \boldsymbol{x}^1, \ (\boldsymbol{y}, \boldsymbol{z}) = (1 - \theta_d^k)(\tilde{\boldsymbol{y}}, \tilde{\boldsymbol{z}}) + \theta_d^k (\boldsymbol{y}^1, \boldsymbol{z}^1) \ \text{ and } \ \omega_p = \omega_d = \omega.$$

Then, we can easily verify that $(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z})$ is a feasible solution of the primal-dual pair of $\mathcal{P}(\theta_p^k, \theta_d^k)$ and $\mathcal{D}(\theta_p^k, \theta_d^k)$, satisfying all the assumptions in (19) of Lemma 4.2. Hence, by Lemma 4.2, we have

$$(\omega)^2 + (\boldsymbol{x}^1)^T \boldsymbol{z}^1 > \delta\omega^*,$$

which contradicts the assumption of the theorem. This completes the proof. ∎

The conclusion of Theorem 4.1 does not necessarily imply the infeasibility of the primal problem $\mathcal{P}$ nor the dual problem $\mathcal{D}$. That is, there may exist a feasible solution $(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z})$ of the primal-dual pair of $\mathcal{P}$ and $\mathcal{D}$ outside of $S(\delta, u)$. Specifically, Theorem 4.1 can not be applied to degenerate cases where both problems $\mathcal{P}$ and $\mathcal{D}$ are feasible but $\mathcal{P}$ or $\mathcal{D}$ has no interior feasible solution. In such cases, there exists no feasible solution $(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}) \in S(\delta, \omega)$ of the primal-dual pair of $\mathcal{P}$ and $\mathcal{D}$ for any small positive δ.

5. Detecting Infeasibility – II

In order to overcome the shortcomings of Theorem 4.1, we need to somewhat modify the algorithm. The modification is done by replacing Step 3 by Step 3' below. It is designed so that once the primal feasibility error $\|A\boldsymbol{x}^k - \boldsymbol{b}\|$ becomes less than or equal to the tolerance ϵ_p (or the dual feasibility error $\|A^T\boldsymbol{y}^k + \boldsymbol{z}^k - \boldsymbol{c}\|$ becomes less than or equal to the tolerance ϵ_p) at some iteration k, the error will never be improved but maintained from then on;

$$\begin{aligned}
&\|A\boldsymbol{x}^j - \boldsymbol{b}\| = \|A\boldsymbol{x}^k - \boldsymbol{b}\| \ \text{ if } \ \|A\boldsymbol{x}^k - \boldsymbol{b}\| \leq \epsilon_p \ \text{ and } j \geq k, \\
&\|A^T\boldsymbol{y}^j + \boldsymbol{z}^j - \boldsymbol{c}\| = \|A^T\boldsymbol{y}^k + \boldsymbol{z}^k - \boldsymbol{c}\| \ \text{ if } \ \|A^T\boldsymbol{y}^k + \boldsymbol{z}^k - \boldsymbol{c}\| \leq \epsilon_p \ \text{ and } j \geq k.
\end{aligned}$$

Step 3': Let $\mu = \beta_1(\boldsymbol{x}^k)^T\boldsymbol{z}^k/n$. In the system (4) of equation, replace $\boldsymbol{A}\boldsymbol{x}^k - \boldsymbol{b}$ by $\boldsymbol{0}$ if $\|\boldsymbol{A}\boldsymbol{x}^k - \boldsymbol{b}\| \leq \epsilon_p$, and replace $\boldsymbol{A}^T\boldsymbol{y}^k + \boldsymbol{z}^k - \boldsymbol{c}$ by $\boldsymbol{0}$ if $\|\boldsymbol{A}^T\boldsymbol{y}^k + \boldsymbol{z}^k - \boldsymbol{c}\| \leq \epsilon_d$. Compute the unique solution $(\boldsymbol{\Delta x}, \boldsymbol{\Delta y}, \boldsymbol{\Delta z})$ at $(\boldsymbol{x}^k, \boldsymbol{y}^k, \boldsymbol{z}^k)$ of the system (4) or its modification mentioned just above when $\|\boldsymbol{A}\boldsymbol{x}^k - \boldsymbol{b}\| \leq \epsilon_p$ and/or $\|\boldsymbol{A}^T\boldsymbol{y}^k + \boldsymbol{z}^k - \boldsymbol{c}\| \leq \epsilon_d$.

It is easily seen that all the relations in (12) through (14) and (18) remain valid for the sequence $\{(\boldsymbol{x}^k, \boldsymbol{y}^k, \boldsymbol{z}^k)\}$ generated by the modified algorithm. Hence, each $(\boldsymbol{x}^k, \boldsymbol{y}^k, \boldsymbol{z}^k)$ is an interior feasible solution of the primal-dual pair of $\mathcal{P}(\theta_p^k, \theta_d^k)$ and $\mathcal{D}(\theta_p^k, \theta_d^k)$, so that Lemma 4.2 also remains valid. The equalities in (11), however, need some modification. For every $\alpha \geq 0$,

$$\begin{aligned}
&\boldsymbol{A}(\boldsymbol{x}^k + \alpha\boldsymbol{\Delta x}) - \boldsymbol{b} = \begin{cases} (1-\alpha)(\boldsymbol{A}\boldsymbol{x}^k - \boldsymbol{b}) & \text{if } \ \|\boldsymbol{A}\boldsymbol{x}^k - \boldsymbol{b}\| > \epsilon_p, \\ \boldsymbol{A}\boldsymbol{x}^k - \boldsymbol{b} & \text{if } \ \|\boldsymbol{A}\boldsymbol{x}^k - \boldsymbol{b}\| \leq \epsilon_p, \end{cases} \\
&\boldsymbol{A}^T(\boldsymbol{y}^k + \alpha\boldsymbol{\Delta y}) + (\boldsymbol{z}^k + \alpha\boldsymbol{\Delta z}) - \boldsymbol{c} \\
&\qquad = \begin{cases} (1-\alpha)(\boldsymbol{A}^T\boldsymbol{y}^k + \boldsymbol{z}^k - \boldsymbol{c}) & \text{if } \ \|\boldsymbol{A}^T\boldsymbol{y}^k + \boldsymbol{z}^k - \boldsymbol{c}\| > \epsilon_d, \\ \boldsymbol{A}^T\boldsymbol{y}^k + \boldsymbol{z}^k - \boldsymbol{c} & \text{if } \ \|\boldsymbol{A}^T\boldsymbol{y}^k + \boldsymbol{z}^k - \boldsymbol{c}\| \leq \epsilon_d. \end{cases}
\end{aligned}$$

Furthermore, we can show that the modified algorithm stops at Step 2 in a finite number of iterations. The proof is omitted here; it is similar but needs some additional arguments to the proof given in Section 3 for the original algorithm.

Let

$$\epsilon_p' = \begin{cases} \epsilon_p / \|\boldsymbol{A}\boldsymbol{x}^1 - \boldsymbol{b}\| & \text{if} \quad \|\boldsymbol{A}\boldsymbol{x}^1 - \boldsymbol{b}\| > 0, \\ +\infty & \text{if} \quad \|\boldsymbol{A}\boldsymbol{x}^1 - \boldsymbol{b}\| = 0, \end{cases}$$
$$\epsilon_d' = \begin{cases} \epsilon_d / \|\boldsymbol{A}^T\boldsymbol{y}^1 + \boldsymbol{z}^1 - \boldsymbol{c}\| & \text{if} \quad \|\boldsymbol{A}^T\boldsymbol{y}^1 + \boldsymbol{z}^1 - \boldsymbol{c}\| > 0, \\ +\infty & \text{if} \quad \|\boldsymbol{A}^T\boldsymbol{y}^1 + \boldsymbol{z}^1 - \boldsymbol{c}\| = 0. \end{cases}$$

Then, the stopping criterion (1) can be rewritten as

$$(\boldsymbol{x}^k)^T \boldsymbol{z}^k \leq \epsilon, \ \theta_p^k \leq \epsilon_p' \ \text{ and } \theta_d^k \leq \epsilon_d'. \tag{20}$$

Along the sequence generated by the modified algorithm, we define

$$\begin{aligned} \delta^k &= \max\{\delta : \delta \boldsymbol{e} \leq \boldsymbol{x}^k, \ \delta \boldsymbol{e} \leq \boldsymbol{z}^k\}, \\ \omega^k &= \|(\boldsymbol{x}^k, \boldsymbol{z}^k)\|_1 = \boldsymbol{e}^T \boldsymbol{x}^k + \boldsymbol{e}^T \boldsymbol{z}^k. \end{aligned}$$

We now assume that the modified algorithm has stopped at an s'th iteration with satisfying the criterion (2) but not (1). Then

$$\|(\boldsymbol{x}^s, \boldsymbol{z}^s)\|_1 \geq \omega^* \text{ and } (\boldsymbol{x}^s)^T \boldsymbol{z}^s \geq \epsilon^*,$$

where ϵ^* denotes the positive constant given in (16), and one of the following four cases occurs:

(a) $\|\boldsymbol{A}\boldsymbol{x}^s - \boldsymbol{b}\| \leq \epsilon_p$ and $\|\boldsymbol{A}^T\boldsymbol{y}^s + \boldsymbol{z}^s - \boldsymbol{c}\| \leq \epsilon_d$ (*i.e.*, $\theta_p^s \leq \epsilon_p'$ and $\theta_d^s \leq \epsilon_d'$).
(b) $\|\boldsymbol{A}\boldsymbol{x}^s - \boldsymbol{b}\| \leq \epsilon_p$ and $\|\boldsymbol{A}^T\boldsymbol{y}^s + \boldsymbol{z}^s - \boldsymbol{c}\| \geq \epsilon_d$ (*i.e.*, $\theta_p^s \leq \epsilon_p'$ and $\theta_d^s \geq \epsilon_d'$).
(c) $\|\boldsymbol{A}\boldsymbol{x}^s - \boldsymbol{b}\| \geq \epsilon_p$ and $\|\boldsymbol{A}^T\boldsymbol{y}^s + \boldsymbol{z}^s - \boldsymbol{c}\| \leq \epsilon_d$ (*i.e.*, $\theta_p^s \geq \epsilon_p'$ and $\theta_d^s \leq \epsilon_d'$).
(d) $\|\boldsymbol{A}\boldsymbol{x}^s - \boldsymbol{b}\| \geq \epsilon_p$ and $\|\boldsymbol{A}^T\boldsymbol{y}^s + \boldsymbol{z}^s - \boldsymbol{c}\| \geq \epsilon_d$ (*i.e.*, $\theta_p^s \geq \epsilon_p'$ and $\theta_d^s \geq \epsilon_d'$).

Let q be the number of the first iteration such that $\theta_p^q \leq \epsilon_p'$ in the cases (a) and (b), and r the first iteration number such that $\theta_d^r \leq \epsilon_d'$ in the cases (a) and (c).

We first deal with the case (a). In this case, we have

$$\begin{aligned} &\|\boldsymbol{b}(\theta_p^q) - \boldsymbol{b}\| \leq \epsilon_p, \ \boldsymbol{A}\boldsymbol{x}^k = \boldsymbol{b}(\theta_p^q) \ \text{ for every } k \geq q, \\ &\|\boldsymbol{c}(\theta_d^r) - \boldsymbol{c}\| \leq \epsilon_d, \ \boldsymbol{A}^T\boldsymbol{y}^k + \boldsymbol{z}^k = \boldsymbol{c}(\theta_d^r) \ \text{ for every } k \geq r. \end{aligned}$$

Letting $\ell = \max\{q, \ r\}$, we see that for every $k \geq \ell$, the modified algorithm works as Mizuno-Todd-Ye's interior-point algorithm [18] applied to the primal-dual pair of the linear programs $\mathcal{P}(\theta_p^q, \theta_d^r)$ and $\mathcal{D}(\theta_p^q, \theta_d^r)$ starting from $(\boldsymbol{x}^\ell, \boldsymbol{y}^\ell, \boldsymbol{z}^\ell)$. Their algorithm is known to reduce the duality gap $(\boldsymbol{x}^k)^T \boldsymbol{z}^k$ by a constant factor which is independent of the location of the iterate $(\boldsymbol{x}^k, \boldsymbol{y}^k, \boldsymbol{z}^k)$. Hence, if we neglect the stopping criterion (2) and continue running the modified algorithm beyond the s'th iteration, we eventually obtain an $(\boldsymbol{x}^k, \boldsymbol{y}^k, \boldsymbol{z}^k)$ satisfying the stopping criterion (1); $(\boldsymbol{x}^k, \boldsymbol{y}^k, \boldsymbol{z}^k)$ gives a desired approximate optimal solution of the primal-dual pair of $\mathcal{P}$ and $\mathcal{D}$.

In the case (b), we obtain that

$$
\begin{aligned}
&\|\boldsymbol{b}(\theta_p^q) - \boldsymbol{b}\| \leq \epsilon_p, \\
&\boldsymbol{A}\boldsymbol{x}^k = \boldsymbol{b}(\theta_p^q) \ \text{ for every } k \geq q, \\
&\|\boldsymbol{A}^T \boldsymbol{y}^s + \boldsymbol{z}^s - \boldsymbol{c}\| = \theta_d^s \|\boldsymbol{A}^T \boldsymbol{y}^1 + \boldsymbol{z}^1 - \boldsymbol{c}\| > \epsilon_d \ (i.e.,\ \theta_d^s > \epsilon_d')
\end{aligned}
$$

The second relation above together with $\boldsymbol{x}^k > \boldsymbol{0}$ $(k \geq q)$ implies that $\boldsymbol{x}^k$ is a feasible solution of a common primal problem $\mathcal{P}(\theta_p^q, 0)$ for every $k \geq q$.

Theorem 5.1. *Suppose that the case (b) occurs. Let $\omega \geq \omega^1$ and $\delta = \min\{\epsilon_d' \delta^1,\ \delta^q\}$. Assume that*

$$\omega^* \geq \frac{\omega^q \omega + (\boldsymbol{x}^1)^T \boldsymbol{z}^1}{\delta}.$$

Then there is no feasible solution $(\tilde{\boldsymbol{y}}, \tilde{\boldsymbol{z}})$ of the problem $\mathcal{D}$ such that $\|\tilde{\boldsymbol{z}}\|_1 \leq \omega$.

Proof: Assume, on the contrary, that there exists a feasible solution $(\tilde{\boldsymbol{y}}, \tilde{\boldsymbol{z}})$ of the problem $\mathcal{D}$ such that $\|\tilde{\boldsymbol{z}}\|_1 \leq \omega$. Let

$$
\begin{aligned}
&\boldsymbol{x} = \boldsymbol{x}^q,\ (\boldsymbol{y}, \boldsymbol{z}) = (1 - \theta_d^s)(\tilde{\boldsymbol{y}}, \tilde{\boldsymbol{z}}) + \theta_d^s(\boldsymbol{y}^1, \boldsymbol{z}^1), \\
&\omega_p = \omega^q \ \text{ and } \ \omega_d = \omega.
\end{aligned}
$$

Then, we can easily verify that $(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z})$ is a feasible solution of the primal-dual pair of $\mathcal{P}(\theta_p^s, \theta_d^s)$ and $\mathcal{D}(\theta_p^s, \theta_d^s)$, satisfying all the assumptions in (19) of Lemma 4.2. Hence,

$$\omega^q \omega + (\boldsymbol{x}^1)^T \boldsymbol{z}^1 > \delta \omega^*,$$

which contradicts the assumption of the theorem. ∎

Similarly, we obtain the following result.

Theorem 5.2. *Suppose that the case (c) occurs. Let $\omega \geq \omega^1$ and $\delta = \min\{\epsilon_p' \delta^1,\ \delta^r\}$. Assume that*

$$\omega^* \geq \frac{\omega^r \omega + (\boldsymbol{x}^1)^T \boldsymbol{z}^1}{\delta}.$$

Then, there is no feasible solution $\tilde{\boldsymbol{x}}$ of the problem $\mathcal{P}$ such that $\|\tilde{\boldsymbol{x}}\|_1 \leq \omega$.

It should be noted that ω^q and δ^q in Theorem 5.1 (ω^r and δ^r in Theorem 5.2) are not known prior to the starting the algorithm. So, to apply Theorems 5.1 or 5.2, we need to adjust the values of ω^* and/or ω during the execution of the algorithm. Finally, we consider the case (d).

Theorem 5.3. *Suppose that the case (d) occurs. Let $\omega \geq \omega^1$ and $\delta = \min\{\epsilon'_p\delta^1, \epsilon'_d\delta^1\}$. Assume that*

$$\omega^* \geq \frac{(\omega)^2 + (\boldsymbol{x}^1)^T \boldsymbol{z}^1}{\delta}.$$

Then, the set

$$S(0, \omega) = \{(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}) \in Q_+ : \|(\boldsymbol{x}, \boldsymbol{z})\|_1 \leq \omega\}$$

contains no feasible solution of the primal-dual pair of $\mathcal{P}$ and $\mathcal{D}$.

Proof: In the case (d), we have

$$\begin{aligned}
&\|\boldsymbol{A}\boldsymbol{x}^s - \boldsymbol{b}\| = \theta^s_p \|\boldsymbol{A}\boldsymbol{x}^1 - \boldsymbol{b}\| > \epsilon_p \ (i.e., \theta^s_p > \epsilon'_p),\\
&\|\boldsymbol{A}^T\boldsymbol{y}^s + \boldsymbol{z}^s - \boldsymbol{c}\| = \theta^s_d \|\boldsymbol{A}^T\boldsymbol{y}^1 + \boldsymbol{z}^1 - \boldsymbol{c}\| > \epsilon_d \ (i.e., \ \theta^s_d > \epsilon'_d).
\end{aligned}$$

Now assume, on the contrary, that there exists a feasible solution $(\tilde{\boldsymbol{x}}, \tilde{\boldsymbol{y}}, \tilde{\boldsymbol{z}})$ of the primal-dual pair of $\mathcal{P}$ and $\mathcal{D}$ such that $\|(\tilde{\boldsymbol{x}}, \tilde{\boldsymbol{z}})\|_1 \leq \omega$. Let

$$\begin{aligned}
&\boldsymbol{x} = (1 - \theta^s_p)\tilde{\boldsymbol{x}} + \theta^s_p \boldsymbol{x}^1, \ (\boldsymbol{y}, \boldsymbol{z}) = (1 - \theta^s_d)(\tilde{\boldsymbol{y}}, \tilde{\boldsymbol{z}}) + \theta^s_d(\boldsymbol{y}^1, \boldsymbol{z}^1),\\
&\omega_p = \omega_d = \omega.
\end{aligned}$$

Then, $(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z})$ is a feasible solution of the primal-dual pair of $\mathcal{P}(\theta^s_p, \theta^s_d)$ and $\mathcal{D}(\theta^s_p, \theta^s_d)$ satisfying all the assumptions in (19) of Lemma 4.2. Hence,

$$(\omega)^2 + (\boldsymbol{x}^1)^T \boldsymbol{z}^1 > \delta\omega^*,$$

which contradicts the assumption of the theorem. ∎

6. Concluding Remarks.

The primal-dual interior-point algorithm for linear programs has been extended to various mathematical programming problems such as convex quadratic programs, convex programs, linear and nonlinear complementarity problems ([4, 6, 8, 10, 20, 21], etc.). It is possible to apply the basic idea of controlling the step length in the way proposed in this paper to infeasible-interior-point algorithms for such problems. Among others, we can easily modify the infeasible-interior-point algorithm using the step length control rule, which has been described in Section 2, so as to adapt it to the linear complementarity problem with an $n \times n$ positive semi-definite matrix $\boldsymbol{M}$: find $(\boldsymbol{x}, \boldsymbol{z}) \in R^{2n}$ such that $(\boldsymbol{x}, \boldsymbol{z}) \geq \boldsymbol{0}$, $\boldsymbol{z} = \boldsymbol{M}\boldsymbol{x} + \boldsymbol{q}$ and $\boldsymbol{x}^T\boldsymbol{z} = 0$. In fact, we define the central trajectory (Megiddo [16]) as the set of solutions $(\boldsymbol{x}, \boldsymbol{z}) > \boldsymbol{0}$ to the system of equations

$$\boldsymbol{z} = \boldsymbol{M}\boldsymbol{x} + \boldsymbol{q} \quad \text{and} \quad \boldsymbol{X}\boldsymbol{z} = \mu\boldsymbol{e}$$

for every $\mu > 0$, the Newton direction at a k'th iterate $(\boldsymbol{x}^k, \boldsymbol{z}^k) > \boldsymbol{0}$ as the unique solution $(\boldsymbol{\Delta x}, \boldsymbol{\Delta z})$ of the system of linear equations

$$\begin{aligned}
&\boldsymbol{\Delta z} - \boldsymbol{M}\boldsymbol{\Delta x} = -\boldsymbol{z}^k + \boldsymbol{M}\boldsymbol{x}^k + \boldsymbol{q}, \\
&\boldsymbol{Z}^k\boldsymbol{\Delta x} + \boldsymbol{X}^k\boldsymbol{\Delta z} = -\boldsymbol{X}^k\boldsymbol{z}^k + (\beta_1(\boldsymbol{x}^k)^T\boldsymbol{z}^k/n)\boldsymbol{e},
\end{aligned}$$

and the neighborhood $\mathcal{N}$ of the central trajectory as

$$\begin{aligned}
\mathcal{N} = \{(\boldsymbol{x}, \boldsymbol{z}) > \boldsymbol{0} \quad : \quad & x_i z_i \geq \gamma\boldsymbol{x}^T\boldsymbol{z}/n \quad (i = 1, 2, \ldots, n) \\
& \boldsymbol{x}^T\boldsymbol{z} \geq \gamma'\|\boldsymbol{z} - \boldsymbol{M}\boldsymbol{x} - \boldsymbol{q}\| \quad \text{or} \quad \|\boldsymbol{z} - \boldsymbol{M}\boldsymbol{x} - \boldsymbol{q}\| \leq \epsilon'\},
\end{aligned}$$

where $0 < \beta_1 < 1$, $0 < \gamma < 1$, $0 < \gamma'$ and $0 < \epsilon'$. Starting from an arbitrary point $(\boldsymbol{x}^1, \boldsymbol{z}^1)$ in $\mathcal{N}$, the algorithm iteratively generates a new point $(\boldsymbol{x}^{k+1}, \boldsymbol{z}^{k+1})$ such that

$$\begin{aligned}
&(\boldsymbol{x}^{k+1}, \boldsymbol{z}^{k+1}) = (\boldsymbol{x}^k, \boldsymbol{z}^k) + \alpha^k(\boldsymbol{\Delta x}, \boldsymbol{\Delta z}) \in \mathcal{N}, \\
&(\boldsymbol{x}^{k+1})^T\boldsymbol{z}^{k+1} \leq (1 - \bar{\alpha}^k(1 - \beta_3))(\boldsymbol{x}^k)^T\boldsymbol{z}^k,
\end{aligned}$$

where $\bar{\alpha}^k$ be the maximum of $\tilde{\alpha}$'s ≤ 1 such that the relations

$$\begin{aligned}
&(\boldsymbol{x}^k, \boldsymbol{z}^k) + \alpha(\boldsymbol{\Delta x}, \boldsymbol{\Delta z}) \in \mathcal{N}, \\
&(\boldsymbol{x}^k + \alpha\boldsymbol{\Delta x})^T(\boldsymbol{z}^k + \alpha\boldsymbol{\Delta z}) \leq (1 - \alpha(1 - \beta_2))(\boldsymbol{x}^k)^T\boldsymbol{z}^k,
\end{aligned}$$

and $0 < \beta_1 < \beta_2 < \beta_3 < 1$. We could show similar results to the ones stated in Sections 3, 4 and 5, but the details are omitted.

We called an infeasible-interior-point algorithm "an exterior point algorithm" in the original manuscript. But many people have pointed out to us that the terminology "exterior" is inappropriate because the algorithm still generates a sequence of points within the interior of the region determined by the nonnegativity constraints and an exterior point algorithm usually means an algorithm that relaxes the nonnegativity constrains (see, for example, [3]). Y. Zhang proposed to use the new name "an infeasible interior-point algorithm." But it may give an impression "an infeasible algorithm" to the readers. To avoid such an impression, we have modified it to "an infeasible-interior-point algorithm."

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