Homotopy Continuation Methods for Nonlinear Complementarity Problems*

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Abstract. A complementarity problem with a continuous mapping $\boldsymbol{f}$ from the n-dimensional Euclidean space R^n into itself can be written as the system of equations

$$\boldsymbol{F}(\boldsymbol{x}, \boldsymbol{y}) = \boldsymbol{0} \text{ and } (\boldsymbol{x}, \boldsymbol{y}) \geq \boldsymbol{0}.$$

Here $\boldsymbol{F}$ is the mapping from R^{2n} into itself defined by

$$\boldsymbol{F}(\boldsymbol{x}, \boldsymbol{y}) = (x_1 y_1, x_2 y_2, \cdots, x_n y_n, \boldsymbol{y} - \boldsymbol{f}(\boldsymbol{x})) \text{ for every } (\boldsymbol{x}, \boldsymbol{y}) \geq \boldsymbol{0}.$$

Under the assumption that the mapping $\boldsymbol{f}$ is a P_0-function, we study various aspects of homotopy continuation methods that trace a trajectory consisting of solutions of the family of systems of equations

$$\boldsymbol{F}(\boldsymbol{x}, \boldsymbol{y}) = t(\boldsymbol{a}, \boldsymbol{b}) \text{ and } (\boldsymbol{x}, \boldsymbol{y}) \geq \boldsymbol{0}$$

until the parameter $t \geq 0$ attains 0. Here $(\boldsymbol{a}, \boldsymbol{b})$ denotes a $2n$-dimensional constant positive vector. We establish the existence of a trajectory which leads to a solution of the problem, and then present a numerical method for tracing the trajectory. We also discuss the global and local convergence of the method.

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1. Introduction

Let R^n denote the n-dimensional Euclidean space. We use the notation R^n_+ for the nonnegative orthant $\{\boldsymbol{x} \in R^n : \boldsymbol{x} \geq \boldsymbol{0}\}$ and R^n_{++} for the positive orthant $\{\boldsymbol{x} \in R^n : \boldsymbol{x} > \boldsymbol{0}\}$ of R^n. The *complementarity problem* $CP[\boldsymbol{f}]$ with respect to a continuous mapping $\boldsymbol{f} : R^n \to R^n$ (see, for example, Cottle [1], Karamardian [6], Kojima [7], Lemke and Howson [15], etc.) is defined to be the problem of finding a $\boldsymbol{z} \in R^{2n}$ such that $\boldsymbol{z} = (\boldsymbol{x}, \boldsymbol{y}) \geq \boldsymbol{0}$, $\boldsymbol{y} = \boldsymbol{f}(\boldsymbol{x})$ and $x_i y_i = 0$ $(i = 1, 2, \cdots, n)$. Under the nonnegativity condition $\boldsymbol{z} = (\boldsymbol{x}, \boldsymbol{y}) \geq \boldsymbol{0}$, the *complementarity condition* $x_i y_i = 0$ $(i = 1, 2, \cdots, n)$ can be rewritten as the condition that the inner product $\boldsymbol{x} \cdot \boldsymbol{y} = \boldsymbol{x}^T \boldsymbol{y}$ is equal to zero. We say that the $CP[\boldsymbol{f}]$ is *linear* if the mapping $\boldsymbol{f}$ is a linear mapping of the form $\boldsymbol{f}(\boldsymbol{x}) = \boldsymbol{M}\boldsymbol{x} + \boldsymbol{q}$ for some $n \times n$ matrix $\boldsymbol{M}$ and $\boldsymbol{q} \in R^n$, and *nonlinear* otherwise. A *feasible solution* is a $\boldsymbol{z} = (\boldsymbol{x}, \boldsymbol{y}) \in R^{2n}$ satisfying the nonnegativity condition $\boldsymbol{z} = (\boldsymbol{x}, \boldsymbol{y}) \geq \boldsymbol{0}$ and the equality $\boldsymbol{y} = \boldsymbol{f}(\boldsymbol{x})$. To distinguish a solution of the $CP[\boldsymbol{f}]$ from a feasible solution, we often call a solution of the $CP[\boldsymbol{f}]$ a *complementary solution.* We use the symbols $S_+[\boldsymbol{f}]$ for the set of all the feasible solutions, and $S_{++}[\boldsymbol{f}]$ for the set of all the strictly positive feasible solutions:

$$\begin{aligned} S_+[\boldsymbol{f}] &= \{\boldsymbol{z} = (\boldsymbol{x}, \boldsymbol{y}) \in R^{2n}_+ \; : \; \boldsymbol{y} = \boldsymbol{f}(\boldsymbol{x})\}, \\ S_{++}[\boldsymbol{f}] &= \{\boldsymbol{z} = (\boldsymbol{x}, \boldsymbol{y}) \in S_+[\boldsymbol{f}] \; : (\boldsymbol{x}, \boldsymbol{y}) > \boldsymbol{0}\}. \end{aligned}$$

This paper studies homotopy continuation methods for nonlinear complementarity problems, which were originally developed for linear programs (Gonzaga [3], Kojima, Mizuno and Yoshise [12], Monteiro and Adler [20], Renegar [23], Vaidya [26], etc.), and then extended to linear complementarity problems (Kojima, Mizuno and Yoshise [13], Megiddo [17]) and nonlinear complementarity problems (Kojima, Mizuno and Noma [10; 11]). See also Jarre [5], Mehrotra and Sun [18], Monteiro and Adler [21], Ye [27] for extensions to quadratic programs. A common basic idea of the algorithms in this class is tracing the path of centers (or analytic centers) of polytopes which leads to solutions. This idea was proposed by Sonnevend [25], and the first polynomial time algorithm in this class was given by Renegar [23]. We also refer to Megiddo [17] who generalized the idea to linear complementarity problems and, in particular, to linear programs in the primal-dual setting.

We describe an outline of the homotopy continuation method for the $CP[\boldsymbol{f}]$. Let $\boldsymbol{X} = \operatorname{diag} \boldsymbol{x}$ denote the $n \times n$ diagonal matrix with the coordinates of a vector $\boldsymbol{x} \in R^n$. Define the mapping $\boldsymbol{F}$ from R^{2n}_+ into $R^n_+ \times R^n$ by

$$\boldsymbol{F}(\boldsymbol{z}) = (\boldsymbol{X}\boldsymbol{y}, \boldsymbol{y} - \boldsymbol{f}(\boldsymbol{x})) \;\text{ for every } \boldsymbol{z} = (\boldsymbol{x}, \boldsymbol{y}) \geq \boldsymbol{0} \tag{1}$$

to rewrite the $CP[\boldsymbol{f}]$ into the system of equations:

$$\boldsymbol{F}(\boldsymbol{z}) = \boldsymbol{0} \;\text{ and }\; \boldsymbol{z} = (\boldsymbol{x}, \boldsymbol{y}) \geq \boldsymbol{0}. \tag{2}$$

Let $\boldsymbol{c} = (\boldsymbol{a}, \boldsymbol{b}) \in R^n_{++} \times R^n$. Consider the family of systems of equations with a nonnegative real parameter t:

$$\boldsymbol{F}(\boldsymbol{z}) = t\boldsymbol{c} \quad \text{and} \quad \boldsymbol{z} = (\boldsymbol{x}, \boldsymbol{y}) \geq \boldsymbol{0}. \tag{3}$$

Obviously, the system (3) with the parameter $t = 0$ coincides with the system (2) or the $CP[\boldsymbol{f}]$. Let

$$C = \{t\boldsymbol{c} : t > 0\}.$$

Under certain assumptions, the system (3) has a unique solution $\boldsymbol{z}(t)$ for each positive t such that $\boldsymbol{z}(t)$ is continuous in the parameter t; hence the set

$$\boldsymbol{F}^{-1}(C) = \{\boldsymbol{z} \in R^{2n}_+ : \boldsymbol{F}(\boldsymbol{z}) = t\boldsymbol{c} \text{ for some } t > 0\} = \{\boldsymbol{z}(t) : t > 0\}$$

forms a trajectory, a one-dimensional curve. Furthermore, $\boldsymbol{z}$(t) leads to a solution of the system (2) as t tends to 0. Suppose that we know a point $\boldsymbol{z}(t^1) \in \boldsymbol{F}^{-1}(C)$ in advance. Thus, if we start from the known point $\boldsymbol{z}(t^1)$ and trace the trajectory $\boldsymbol{F}^{-1}(C)$ until the parameter t attains zero, we get a solution of the system (2) or a complementary solution of the $CP[\boldsymbol{f}]$.

There are several questions arising from the continuation method described above. Theoretically, we have to establish the existence of a trajectory consisting of solutions of the system (3). We have to study the limiting behavior of the trajectory as the parameter t approaches zero. In addition, we need to show how to prepare an initial point $\boldsymbol{z}(t^1) \in \boldsymbol{F}^{-1}(C)$ as well as how to trace the trajectory $\boldsymbol{F}^{-1}(C)$ numerically. Global and local convergence of the method should be discussed too. These questions have been answered partially for some special cases.

We first consider the case where $\boldsymbol{f}$ is a linear mapping, $\boldsymbol{f}(\boldsymbol{x}) = \boldsymbol{M}\boldsymbol{x} + \boldsymbol{q}$, with a positive semi-definite matrix $\boldsymbol{M}$, i.e., $\boldsymbol{x} \cdot (\boldsymbol{M}\boldsymbol{x}) \geq 0$ for every $\boldsymbol{x} \in R^n$. As a special case of (3), consider the family of systems of equations with a nonnegative real parameter t:

$$\boldsymbol{F}(\boldsymbol{z}) = t(\boldsymbol{e}, \boldsymbol{0}) \quad \text{and} \quad \boldsymbol{z} = (\boldsymbol{x}, \boldsymbol{y}) \geq \boldsymbol{0}. \tag{4}$$

Here $\boldsymbol{e} = (1, \cdots, 1) \in R^n$. Suppose that the set $S_{++}[\boldsymbol{f}]$ of all the strictly positive feasible solutions of the $CP[\boldsymbol{f}]$ is nonempty. Then the set of the solutions $\boldsymbol{z}$ of the system (4) with the positive t's forms a trajectory $\{\boldsymbol{z}(t) : t > 0\}$ which converges to a complementary solution of the $CP[\boldsymbol{f}]$ as t tends to zero (Megiddo [17]). In this case the trajectory can be regarded as a generalization of the path of centers of a system of linear inequalities. The algorithm given by Kojima, Mizuno and Yoshise [13] computes a complementary solution of the $CP[\boldsymbol{f}]$ by tracing the path of centers numerically.

The system (4) can be rewritten as

$$\boldsymbol{y} = \boldsymbol{M}\boldsymbol{x} + \boldsymbol{q}, \ \boldsymbol{X}\boldsymbol{y} = t\boldsymbol{e} \quad \text{and} \quad \boldsymbol{z} = (\boldsymbol{x}, \boldsymbol{y}) \geq \boldsymbol{0}.$$

Hence the solution $\boldsymbol{z}(t)$ of the system (4) is restricted to running in the relative interior $S_{++}[\boldsymbol{f}]$ of the fixed feasible region $S_+[\boldsymbol{f}]$. This lacks the flexibility in choosing an initial point. Theoretically, we can construct an artificial problem which has an initial point $\boldsymbol{z}^0 \in S_{++}[\boldsymbol{f}]$ sufficiently close to the path of centers (See Section 6 of [13]). However the magnitude of such a theoretical initial point is too large for implementation on computers (Lustig [16], Mizuno, Yoshise and Kikuchi [19]). The family of systems of equations (3) gives us more freedom in choosing initial points than the family (4).

We consider now more general nonlinear cases. If the system (3) has a solution for every $t > 0$, we must have

$$t\boldsymbol{b} \in B_{++}[\boldsymbol{f}] \equiv \{\boldsymbol{u} \in R^n : \boldsymbol{u} = \boldsymbol{y} - \boldsymbol{f}(\boldsymbol{x}) \text{ for some } (\boldsymbol{x}, \boldsymbol{y}) > \mathbf{0}\}$$

for every $t > 0$. Hence it is necessary to take a vector $\boldsymbol{b} \in R^n$ such that $t\boldsymbol{b} \in B_{++}[\boldsymbol{f}]$ for every $t > 0$. When we describe a numerical method in Section 5, we will assume $\boldsymbol{b} > \mathbf{0}$ to meet this necessary condition. In fact, by the definition of the sets $S_{++}[\boldsymbol{f}]$ and $B_{++}[\boldsymbol{f}]$, we see that if the $CP[\boldsymbol{f}]$ has a strictly positive feasible solution, i.e., $S_{++}[\boldsymbol{f}] \neq \emptyset$, then there exists an $(\tilde{\boldsymbol{x}}, \tilde{\boldsymbol{y}}) > \mathbf{0}$ such that $\mathbf{0} = \tilde{\boldsymbol{y}} - \boldsymbol{f}(\tilde{\boldsymbol{x}})$; hence

$$R^n_+ \subset \{\boldsymbol{u} \in R^n_+ : \boldsymbol{u} = \boldsymbol{y} - \boldsymbol{f}(\tilde{\boldsymbol{x}}) \text{ for some } \boldsymbol{y} \in R^n_{++}\} \subset B_{++}[\boldsymbol{f}] \; .$$

This ensures that $t\boldsymbol{b} \in B_{++}[\boldsymbol{f}]$ for every $t > 0$. It should be noted that even in this case, we can start from any $\boldsymbol{x}^1 \in R^n_{++}$ in the $\boldsymbol{x}$-space by taking appropriate $\boldsymbol{y}^1 \in R^n_{++}$ and $\boldsymbol{c} = (\boldsymbol{a}, \boldsymbol{b}) \in R^n_{++}$ such that $\boldsymbol{a} = (x^1_1 y^1_1, x^1_2 y^1_2, \cdots, x^1_n y^1_n)$ and $\boldsymbol{b} = \boldsymbol{y}^1 - \boldsymbol{f}(\boldsymbol{x}^1) > \mathbf{0}$. This flexibility in choosing initial points is very important especially when we apply the continuation method with the use of the family (3) to nonlinear problems where finding a feasible solution is generally as difficult as solving them.

Kojima, Mizuno and Noma [10; 11] presented two conditions to ensure the existence of a trajectory consisting of solutions of the family of systems of equations (3):

Condition 1.1. (Kojima, Mizuno and Noma [10])
The mapping $\boldsymbol{f}$ is a uniform P-function, i.e., there exists a positive number γ such that

$$\max_i \, (x_i - y_i)(f_i(\boldsymbol{x}) - f_i(\boldsymbol{y})) \geq \gamma \|\boldsymbol{x} - \boldsymbol{y}\|^2 \text{ for every } \boldsymbol{x}, \boldsymbol{y} \in R^n.$$

Condition 1.2. (Kojima, Mizuno and Noma [11])

(i) The mapping $\boldsymbol{f}$ is a monotone function, i.e.,

$$(\boldsymbol{x} - \boldsymbol{y}) \cdot (\boldsymbol{f}(\boldsymbol{x}) - \boldsymbol{f}(\boldsymbol{y})) \geq 0 \text{ for every } \boldsymbol{x}, \boldsymbol{y} \in R^n.$$

(ii) The set $S_{++}[\boldsymbol{f}]$ of all the strictly positive feasible solutions of the $CP[\boldsymbol{f}]$ is nonempty.

Remark 1.3. When $\boldsymbol{f}$ is a linear mapping from R^n into itself with an $n \times n$ matrix $\boldsymbol{M}$, it is monotone if and only if $\boldsymbol{M}$ is a positive semi-definite matrix, and a uniform P-function if and only if $\boldsymbol{M}$ is a P-matrix, i.e., all the principal minors of $\boldsymbol{M}$ are positive (Fiedler and Pták [2]). It is well-known that the Karush-Kuhn-Tucker optimality conditions for linear and convex quadratic programs can be formulated as a positive semi-definite linear complementarity problem.

Conditions 1.1 and 1.2 above can be unified as follows:

Lemma 1.4. *If Condition 1.1 or 1.2 holds then Condition 1.5 does.*

Condition 1.5.

(i) $\boldsymbol{f}$ is a P_0-function, i.e., for every $\boldsymbol{x}, \boldsymbol{y} \in R^n$ with $\boldsymbol{x} \neq \boldsymbol{y}$, there is an index i such that

$$x_i - y_i \neq 0 \quad \text{and} \quad (x_i - y_i)(f_i(\boldsymbol{x}) - f_i(\boldsymbol{y})) \geq 0.$$

(ii) The set $S_{++}[\boldsymbol{f}]$ of all the strictly positive feasible solutions is nonempty.

(iii) The set

$$\boldsymbol{F}^{-1}(D) = \{\boldsymbol{z} = (\boldsymbol{x}, \boldsymbol{y}) \in R^{2n}_{+} : \boldsymbol{F}(\boldsymbol{z}) \in D\}$$

is bounded for every compact (i.e., bounded and closed as a subset of R^{2n}) subset D of $R^n_{+} \times B_{++}[\boldsymbol{f}]$.

Remark 1.6. As we have already seen, $R^n_{+} \subset B_{++}[\boldsymbol{f}]$ follows from (ii) of Condition 1.5. Hence, Condition 1.5 implies that the set

$$\boldsymbol{F}^{-1}(D) = \{\boldsymbol{z} = (\boldsymbol{x}, \boldsymbol{y}) \in R^{2n}_{+} : \boldsymbol{F}(\boldsymbol{z}) \in D\}$$

is bounded for every bounded subset D of R^{2n}_{+}. We will often use this fact later.

After listing in Section 2 some symbols and notation, which will be used throughout this paper, we give a proof of Lemma 1.4 in Section 3. In Section 4 we show some basic properties of the mapping $\boldsymbol{F} : R^{2n}_{+} \rightarrow R^n_{+} \times R^n$ defined by (1). Specifically, we establish under Condition 1.5 the existence of the unique trajectory $\boldsymbol{F}^{-1}(C)$ leading to solutions of the $CP[\boldsymbol{f}]$. This result is an extension of the results given by Kojima, Mizuno and Noma [10; 11]. In Section 5 we present an algorithm for tracing the trajectory. This algorithm is a modification and extension of the primal-dual algorithm given by Kojima, Mizuno and Yoshise [12] for linear programs. See also the papers [13; 20; 21]. Although we can apply the algorithm to linear complementarity problems, the main emphasis will be placed on nonlinear cases. In Section 6 we show the global convergence property of the algorithm. In Section 7 we discuss the local convergence property of the algorithm under a nondegeneracy condition, and present a modified algorithm which has a locally quadratic convergence property.

2. Symbols and Notation

R^n : the n-dimensional Euclidean space.
$R^n_+ = \{\boldsymbol{x} \in R^n : \boldsymbol{x} \geq \boldsymbol{0}\}$: the nonnegative orthant of R^n.
$R^n_{++} = \{\boldsymbol{x} \in R^n : \boldsymbol{x} > \boldsymbol{0}\}$: the positive orthant of R^n.
$\boldsymbol{e} = (1, \cdots, 1) \in R^n$.
$\boldsymbol{f}$: a continuous mapping from R^n into itself.
$\boldsymbol{X} = \operatorname{diag} \boldsymbol{x}$: the $n \times n$ diagonal matrix with the coordinates of a vector $\boldsymbol{x} \in R^n$.
$\boldsymbol{F}(\boldsymbol{z}) = (\boldsymbol{X}\boldsymbol{y}, \boldsymbol{y} - \boldsymbol{f}(\boldsymbol{x}))$ for every $\boldsymbol{z} = (\boldsymbol{x}, \boldsymbol{y}) \in R^{2n}_+$.
$CP[\boldsymbol{f}]$ (the complementarity problem) :

$$\text{Find a } \boldsymbol{z} = (\boldsymbol{x}, \boldsymbol{y}) \in R^{2n} \ \text{ such that } \ \boldsymbol{F}(\boldsymbol{z}) = \boldsymbol{0}, \ \boldsymbol{z} \geq \boldsymbol{0}.$$

$S_+[\boldsymbol{f}] = \{\boldsymbol{z} = (\boldsymbol{x}, \boldsymbol{y}) \in R^{2n}_+ : \boldsymbol{y} = \boldsymbol{f}(\boldsymbol{x})\}$: the feasible region of the $CP[\boldsymbol{f}]$.
$S_{++}[\boldsymbol{f}] = \{\boldsymbol{z} = (\boldsymbol{x}, \boldsymbol{y}) \in R^{2n}_{++} : \boldsymbol{y} = \boldsymbol{f}(\boldsymbol{x})\}$.
$B_{++}[\boldsymbol{f}] = \{\boldsymbol{u} \in R^n : \boldsymbol{u} = \boldsymbol{y} - \boldsymbol{f}(\boldsymbol{x}) \ \text{ for some } \ \boldsymbol{z} = (\boldsymbol{x}, \boldsymbol{y}) > \boldsymbol{0}\}$.
$\boldsymbol{c} = (\boldsymbol{a}, \boldsymbol{b}) \in R^n_{++} \times B_{++}[\boldsymbol{f}]$. It will be assumed that $\boldsymbol{c} \in R^{2n}_{++}$ and $\|\boldsymbol{c}\| = 1$.
$C = \{t\boldsymbol{c} : t > 0\}$.
$U \subset R^{2n}_{++} \cup \{\boldsymbol{0}\}$: a closed convex cone whose interior contains the half-line C.
$U_1 = \{\boldsymbol{u} \in U : \boldsymbol{c} \cdot \boldsymbol{u} = 1\}$.
$U(t) = \{\boldsymbol{u} \in U : \boldsymbol{c} \cdot \boldsymbol{u} \leq t\}$ $(t \geq 0)$.
$\bar{\beta} \in (0, \ 1)$, $\phi \in (0, \ \infty)$: constant scalars such that $(1 + \phi)\bar{\beta} < 1$.

3. Proof of Lemma 1.4

First we deal with the case where the mapping $\boldsymbol{f}$ satisfies Condition 1.1. The statement (i) of Condition 1.5 directly follows from the definition of a uniform P-function. It has been shown in [10] that $\boldsymbol{F}$ maps R^{2n}_+ onto $R^n_+ \times R^n$ homeomorphically. In particular there exists an $(\bar{\boldsymbol{x}}, \bar{\boldsymbol{y}}) \in R^{2n}_+$ such that $\boldsymbol{F}(\bar{\boldsymbol{x}}, \bar{\boldsymbol{y}}) = (\boldsymbol{e}, \boldsymbol{0})$. Recall that $\boldsymbol{e} = (1, \cdots, 1) \in R^n$. Since $\bar{x}_i \bar{y}_i = 1$ $(i = 1, 2, \cdots, n)$, we have $(\bar{\boldsymbol{x}}, \bar{\boldsymbol{y}}) \in R^{2n}_{++}$; hence $(\bar{\boldsymbol{x}}, \bar{\boldsymbol{y}}) \in S_{++}[\boldsymbol{f}]$. Thus we have shown (ii) of Condition 1.5, i.e., $S_{++}[\boldsymbol{f}] \neq \emptyset$, or equivalently, $\boldsymbol{0} \in B_{++}[\boldsymbol{f}]$. By a similar argument, we can easily show that $B_{++}[\boldsymbol{f}] = R^n$. If D is a compact subset of $R^n_+ \times R^n$, then $\boldsymbol{F}^{-1}(D)$ is also compact since $\boldsymbol{F}$ is a homeomorphism. Thus, (iii) of Condition 1.5 is satisfied.

Now we consider the case where the mapping $\boldsymbol{f}$ satisfies Condition 1.2. The statements (i) and (ii) of Condition 1.5 follow immediately. To show (iii) of Condition 1.5, assume, on the contrary, that the set

$$E = \boldsymbol{F}^{-1}(D) = \{(\boldsymbol{x}, \boldsymbol{y}) \in R^{2n}_+ : \boldsymbol{F}(\boldsymbol{x}, \boldsymbol{y}) \in D\}$$

is unbounded for some compact subset D of $R^n_+ \times B_{++}[\boldsymbol{f}]$. Then we can take a sequence of points $\{(\boldsymbol{x}^k, \boldsymbol{y}^k) \in E : k = 1, 2, \cdots\}$ such that

$$\lim_{k\to\infty} \|(\boldsymbol{x}^k, \boldsymbol{y}^k)\| = \infty$$

and

$$\lim_{k\to\infty} (\boldsymbol{y}^k - \boldsymbol{f}(\boldsymbol{x}^k)) = \bar{\boldsymbol{u}} \text{ for some } \bar{\boldsymbol{u}} \in B_{++}[\boldsymbol{f}].$$

Since $B_{++}[\boldsymbol{f}]$ is an open subset of R^n, we can find a $\tilde{\boldsymbol{u}} \in B_{++}[\boldsymbol{f}]$ such that

$$\boldsymbol{y}^k - \boldsymbol{f}(\boldsymbol{x}^k) \geq \tilde{\boldsymbol{u}}$$

for every sufficiently large k. By the definition of the set $B_{++}[\boldsymbol{f}]$, there is an $(\tilde{\boldsymbol{x}}, \tilde{\boldsymbol{y}}) \in R^{2n}_{++}$ satisfying $\tilde{\boldsymbol{y}} - \boldsymbol{f}(\tilde{\boldsymbol{x}}) = \tilde{\boldsymbol{u}}$. Furthermore, $(\boldsymbol{X}^k\boldsymbol{y}^k, \boldsymbol{y}^k - \boldsymbol{f}(\boldsymbol{x}^k))$ lies in the bounded set D for every k, where $\boldsymbol{X}^k = \text{diag}\, \boldsymbol{x}^k$. So we can find positive numbers η and ζ such that

$$\boldsymbol{x}^k \cdot \boldsymbol{y}^k \leq \eta \text{ and } \tilde{\boldsymbol{x}} \cdot (\boldsymbol{y}^k - \boldsymbol{f}(\boldsymbol{x}^k) - \tilde{\boldsymbol{u}} + \tilde{\boldsymbol{y}}) \leq \zeta \quad (k = 1, 2, \cdots).$$

Hence, for every sufficiently large k, we have

$$\begin{array}{rcl}
0 & \leq & (\boldsymbol{x}^k - \tilde{\boldsymbol{x}}) \cdot (\boldsymbol{f}(\boldsymbol{x}^k) - \boldsymbol{f}(\tilde{\boldsymbol{x}})) \ \ \text{(by (i) of Condition 1.2)} \\
& = & (\boldsymbol{x}^k - \tilde{\boldsymbol{x}}) \cdot (\boldsymbol{y}^k - (\boldsymbol{y}^k - \boldsymbol{f}(\boldsymbol{x}^k) - \tilde{\boldsymbol{u}} + \tilde{\boldsymbol{y}})) \ \ (\text{by } \ \tilde{\boldsymbol{y}} - \tilde{\boldsymbol{u}} = \boldsymbol{f}(\tilde{\boldsymbol{x}}) \ ) \\
& = & \boldsymbol{x}^k \cdot \boldsymbol{y}^k - \tilde{\boldsymbol{x}} \cdot \boldsymbol{y}^k - \boldsymbol{x}^k \cdot (\boldsymbol{y}^k - \boldsymbol{f}(\boldsymbol{x}^k) - \tilde{\boldsymbol{u}} + \tilde{\boldsymbol{y}}) + \tilde{\boldsymbol{x}} \cdot (\boldsymbol{y}^k - \boldsymbol{f}(\boldsymbol{x}^k) - \tilde{\boldsymbol{u}} + \tilde{\boldsymbol{y}}) \\
& \leq & \boldsymbol{x}^k \cdot \boldsymbol{y}^k - \tilde{\boldsymbol{x}} \cdot \boldsymbol{y}^k - \boldsymbol{x}^k \cdot \tilde{\boldsymbol{y}} + \tilde{\boldsymbol{x}} \cdot (\boldsymbol{y}^k - \boldsymbol{f}(\boldsymbol{x}^k) - \tilde{\boldsymbol{u}} + \tilde{\boldsymbol{y}}) \\
& & \qquad\qquad (\text{since } \ \boldsymbol{x}^k \geq \boldsymbol{0} \ \text{ and } \ \boldsymbol{y}^k - \boldsymbol{f}(\boldsymbol{x}^k) - \tilde{\boldsymbol{u}} \geq \boldsymbol{0} \ ) \\
& \leq & \eta - \tilde{\boldsymbol{x}} \cdot \boldsymbol{y}^k - \boldsymbol{x}^k \cdot \tilde{\boldsymbol{y}} + \zeta.
\end{array}$$

Thus we have obtained

$$\tilde{\boldsymbol{x}} \cdot \boldsymbol{y}^k + \boldsymbol{x}^k \cdot \tilde{\boldsymbol{y}} \leq \eta + \zeta$$

for every sufficiently large k. Since $(\boldsymbol{x}^k, \boldsymbol{y}^k) \in R^{2n}_+ \ \ (k = 1, 2, \cdots)$, the inequality above ensures that the bounded set $\{(\boldsymbol{x}, \boldsymbol{y}) \in R^{2n}_+ : \tilde{\boldsymbol{x}} \cdot \boldsymbol{y} + \boldsymbol{x} \cdot \tilde{\boldsymbol{y}} \leq \eta + \zeta\}$ contains $(\boldsymbol{x}^k, \boldsymbol{y}^k)$ for every sufficiently large k. But this contradicts the fact that $\lim_{k\to\infty} \|(\boldsymbol{x}^k, \boldsymbol{y}^k)\| = \infty$. This completes the proof of Lemma 1.4.

4. The Existence of the Trajectory $\boldsymbol{F}^{-1}(C)$

In the remainder of the paper we assume Condition 1.5. The main assertion of this section is Theorem 4.4, which establishes that the set $\boldsymbol{F}^{-1}(C)$ consisting of the solutions of the system (3) for all positive t forms a trajectory leading to solutions of the $CP[\boldsymbol{f}]$. This result will give a theoretical basis to the homotopy continuation method described in Section 5. To prove the theorem, we need three lemmas. The first two lemmas ensure the existence and uniqueness of a solution of the system of equations

$$\boldsymbol{F}(\boldsymbol{z}) = (\boldsymbol{a}, \boldsymbol{b}) \text{ and } \boldsymbol{z} = (\boldsymbol{x}, \boldsymbol{y}) \in R^{2n}_{+} \tag{5}$$

for every $(\boldsymbol{a}, \boldsymbol{b}) \in R^n_{++} \times B_{++}[\boldsymbol{f}]$, where

$$B_{++}[\boldsymbol{f}] = \{\boldsymbol{u} \in R^n : \boldsymbol{u} = \boldsymbol{y} - \boldsymbol{f}(\boldsymbol{x}) \text{ for some } (\boldsymbol{x}, \boldsymbol{y}) \in R^{2n}_{++}\}.$$

Lemma 4.1. *The mapping* $\boldsymbol{F}$ *is one-to-one on* R^{2n}_{++}.

Proof: Assume on the contrary that $\boldsymbol{F}(\boldsymbol{x}^1, \boldsymbol{y}^1) = \boldsymbol{F}(\boldsymbol{x}^2, \boldsymbol{y}^2)$ for some distinct $(\boldsymbol{x}^1, \boldsymbol{y}^1), (\boldsymbol{x}^2, \boldsymbol{y}^2) \in R^{2n}_{++}$. Then

$$\boldsymbol{f}(\boldsymbol{x}^1) - \boldsymbol{f}(\boldsymbol{x}^2) = \boldsymbol{y}^1 - \boldsymbol{y}^2 \text{ and } x^1_i y^1_i = x^2_i y^2_i > 0 \ \ (i = 1, 2, \cdots, n).$$

Since the mapping $\boldsymbol{f}$ is a P_0-function, we can find an index k such that

$$x^1_k \neq x^2_k \text{ and } 0 \leq (x^1_k - x^2_k)(f_k(\boldsymbol{x}^1) - f_k(\boldsymbol{x}^2)) = (x^1_k - x^2_k)(y^1_k - y^2_k).$$

We may assume without loss of generality that $x^1_k > x^2_k$. Then the inequality above implies that $y^1_k \geq y^2_k$. This contradicts the equality $x^1_k y^1_k = x^2_k y^2_k > 0$. ∎

Lemma 4.2. *The system (5) has a solution for every* $(\boldsymbol{a}, \boldsymbol{b}) \in R^n_{+} \times B_{++}[\boldsymbol{f}]$.

Proof: Let $(\boldsymbol{a}, \boldsymbol{b}) \in R^n_{+} \times B_{++}[\boldsymbol{f}]$. It follows from $\boldsymbol{b} \in B_{++}[\boldsymbol{f}]$ that $\hat{\boldsymbol{y}} - \boldsymbol{f}(\hat{\boldsymbol{x}}) = \boldsymbol{b}$ for some $(\hat{\boldsymbol{x}}, \hat{\boldsymbol{y}}) \in R^{2n}_{++}$. Let $\hat{\boldsymbol{a}} = (\hat{x}_1\hat{y}_1, \hat{x}_2\hat{y}_2, \cdots, \hat{x}_n\hat{y}_n) \in R^n_{++}$. Now we consider the family of systems of equations with the parameter $t \in [0, 1]$:

$$\boldsymbol{F}(\boldsymbol{x}, \boldsymbol{y}) = ((1-t)\hat{\boldsymbol{a}} + t\boldsymbol{a}, \boldsymbol{b}) \text{ and } (\boldsymbol{x}, \boldsymbol{y}) \in R^{2n}_{+}. \tag{6}$$

Let $\bar{t} \leq 1$ be the supremum of $\hat{t}$'s such that the system (6) has a solution for every $t \in [0, \hat{t}\,]$. Then there exists a sequence $\{(\boldsymbol{x}^k, \boldsymbol{y}^k, t^k)\}$ of solutions of the system (6) such that $\lim_{k\to\infty} t^k = \bar{t}$. Since the right-hand side $((1-t)\hat{\boldsymbol{a}} + t\boldsymbol{a}, \boldsymbol{b})$ of the system (6) lies in the compact convex subset $D = \{((1-t)\hat{\boldsymbol{a}} + t\boldsymbol{a}, \boldsymbol{b}) : t \in [0, 1]\}$ of $R^n_{+} \times B_{++}[\boldsymbol{f}]$ for all $t \in [0, 1]$, (iii) of Condition 1.5 ensures that the sequence $\{(\boldsymbol{x}^k, \boldsymbol{y}^k)\}$ is bounded. Hence we may assume that it converges to some $(\bar{\boldsymbol{x}}, \bar{\boldsymbol{y}})$. By the continuity of the mapping $\boldsymbol{F}$, the point $(\bar{\boldsymbol{x}}, \bar{\boldsymbol{y}}, \bar{t})$ satisfies the system (6). Hence if $\bar{t} = 1$ then the desired result follows. Assume on the contrary that $\bar{t} < 1$. Then we have $\bar{x}_i\bar{y}_i = (1-\bar{t})\hat{a}_i + \bar{t}a_i > 0$ for every $i = 1, 2, \cdots, n$. Hence $(\bar{\boldsymbol{x}}, \bar{\boldsymbol{y}}) \in R^{2n}_{++}$. It follows from Lemma 4.1 that the mapping $\boldsymbol{F}$ is a local homeomorphism at $(\bar{\boldsymbol{x}}, \bar{\boldsymbol{y}})$. (See the domain invariance theorem in Schwartz [24].) Hence the system (6) has a solution for every t sufficiently close to $\bar{t}$. This contradicts the definition of $\bar{t}$. ∎

Lemma 4.3.

(i) $R^n_+ \subset B_{++}[\boldsymbol{f}]$.

(ii) *$\boldsymbol{F}$ maps R^{2n}_{++} onto $R^n_{++} \times B_{++}[\boldsymbol{f}]$ homeomorphically.*

Proof: As we have already seen in Section 1, the assertion (i) follows from (ii) of Condition 1.5. We will show the assertion (ii). By the definition, we immediately see $\boldsymbol{F}(R^{2n}_{++}) \subset R^n_{++} \times B_{++}[\boldsymbol{f}]$, and $R^n_{++} \times B_{++}[\boldsymbol{f}] \subset \boldsymbol{F}(R^{2n}_{++})$ by Lemma 4.2. Hence $\boldsymbol{F}$ maps R^{2n}_{++} onto $R^n_{++} \times B_{++}[\boldsymbol{f}]$. By Lemma 4.1, the continuous mapping $\boldsymbol{F}$ is one-to-one on the open subset R^{2n}_{++} of R^{2n}. Thus (ii) follows from the domain invariance theorem (see Schwartz [24]). ∎

We remark here that if a continuous mapping $\boldsymbol{f}$ satisfies Condition 1.1 then $\boldsymbol{F}$ maps R^{2n}_+ onto $R^n_+ \times R^n$ homeomorphically (Kojima, Mizuno and Noma [10]). Now we are ready to establish the existence of the trajectory $\boldsymbol{F}^{-1}(C)$ consisting of solutions of the system (3) for all positive t.

Theorem 4.4. *Let $\boldsymbol{c} = (\boldsymbol{a}, \boldsymbol{b}) \in R^n_{++} \times R^n_+$, and $C = \{t\boldsymbol{c} : t > 0\}$.*

(i) *For every $t > 0$, the system (3) has a unique solution $\boldsymbol{z}(t)$, which is continuous in t; hence the set $\boldsymbol{F}^{-1}(C) = \{\boldsymbol{z}(t) : t > 0\}$ forms a trajectory.*

(ii) *For every $t^0 > 0$, the subtrajectory $\{\boldsymbol{z}(t) : 0 < t < t^0\}$ is bounded; hence there is at least one limiting point of $\boldsymbol{z}(t)$ as $t \to 0$.*

(iii) *Every limiting point of $\boldsymbol{z}(t)$ as $t \to 0$ is a complementary solution of the $CP[\boldsymbol{f}]$.*

(iv) *If $\boldsymbol{f}$ is a linear mapping of the form $\boldsymbol{f}(\boldsymbol{x}) = \boldsymbol{M}\boldsymbol{x} + \boldsymbol{q}$, then $\boldsymbol{z}(t)$ converges to a solution of the $CP[\boldsymbol{f}]$ as $t \to 0$.*

Proof: By (i) of Lemma 4.3, we first observe that

$$t\boldsymbol{c} \in C \subset R^n_{++} \times R^n_+ \subset R^n_{++} \times B_{++}[\boldsymbol{f}]$$

for every $t > 0$. Hence the assertion (i) follows from (ii) of Lemma 4.3. If we take $D = \{t\boldsymbol{c} : 0 < t < t^0\}$, we see by Remark 1.6 that the set

$$\boldsymbol{F}^{-1}(D) = \{\boldsymbol{z} \in R^{2n}_{++} : \boldsymbol{F}(\boldsymbol{z}) \in D\} = \{\boldsymbol{z}(t) : 0 < t < t^0\}$$

is bounded. Thus we obtain (ii). By the continuity of the mapping $\boldsymbol{F}$, if $\boldsymbol{z}$ is a limiting point of $\boldsymbol{z}(t)$ as $t \to 0$, we have $\boldsymbol{F}(\boldsymbol{z}) = \boldsymbol{0}$ and $\boldsymbol{z} \geq \boldsymbol{0}$; hence $\boldsymbol{z}$ is a complementary solution of the $CP[\boldsymbol{f}]$. Thus we have shown (iii). Finally, to see the assertion (iv), we will utilize some result on real algebraic varieties. We call a subset V of R^m a real algebraic variety if there exist a finite number of polynomials g_i $(i = 1, 2, \cdots, k)$ such that

$$V = \{\boldsymbol{x} \in R^m : g_i(\boldsymbol{x}) = 0 \ (i = 1, 2, \cdots, k)\}.$$

We know that a real algebraic variety has a triangulation (see, for example, Hironaka [4]). That is, it is homeomorphic to a locally finite simplicial complex. Let

$$V = \{(\boldsymbol{x}, \boldsymbol{y}, t) \in R^{2n+1} : \boldsymbol{y} = \boldsymbol{M}\boldsymbol{x} + \boldsymbol{q} + t\boldsymbol{b},\ x_i y_i = t a_i\ (i = 1, 2, \cdots, n)\}.$$

Obviously, the set V is a real algebraic variety, so it has a triangulation. Let $\bar{\boldsymbol{z}} = (\bar{\boldsymbol{x}}, \bar{\boldsymbol{y}})$ be a limiting point of $\boldsymbol{z}(t)$ as $t \to 0$. Then the point $\boldsymbol{v} = (\bar{\boldsymbol{x}}, \bar{\boldsymbol{y}}, 0)$ lies in V. Since the triangulation of V is locally finite, we can find a sequence $\{t^p > 0\}$ and a subset σ of V which is homeomorphic to a one-dimensional simplex such that

$$\lim_{p\to\infty} t^p = 0\ ,\ \lim_{p\to\infty} \boldsymbol{z}(t^p) = \bar{\boldsymbol{z}}\ ,\ \boldsymbol{z}(t^p) \in \sigma\ (p = 1, 2, \cdots\ ).$$

But we know that $V \cap R^{2n+1}_{++}$ coincides with the one-dimensional curve $\{(\boldsymbol{z}(t), t) : t > 0\}$. Thus, the subset $\{(\boldsymbol{z}(t), t) : t^{p+1} \leq t \leq t^p\}$ of the curve must be contained in the set σ for every p, since otherwise σ is not arcwise connected. This ensures that $\boldsymbol{z}(t)$ converges to $\bar{\boldsymbol{z}}$ as $t \to 0$. ∎

Remark 4.5. One of the referees suggested another proof of the assertion (iv) of the theorem using the well-known result that every real algebraic variety contains only finitely many connected components. Indeed, for every $\epsilon > 0$, the algebraic variety $V \cap \{(\boldsymbol{x}, \boldsymbol{y}, t) \in R^{2n+1} : \|(\boldsymbol{x}, \boldsymbol{y}, t) - \boldsymbol{v}\|^2 \leq \epsilon^2\}$ has finitely many connected components. It follows that $\boldsymbol{z}(t)$ must be within distance ϵ of $\boldsymbol{v}$ for all sufficiently small $t > 0$.

5. A Numerical Method for Tracing the Trajectory $\boldsymbol{F}^{-1}(C)$

In the previous section, we have shown the existence of the trajectory $\boldsymbol{F}^{-1}(C)$ leading to solutions of the $CP[\boldsymbol{f}]$ for every $\boldsymbol{c} \in R_{++} \times R_{+}$ and $C = \{t\boldsymbol{c} : t > 0\}$. In general, the trajectory $\boldsymbol{F}^{-1}(C)$ is nonlinear, so that exact tracing is difficult even if we know an initial point on the trajectory. Of course, exact tracing is not necessary since our aim is only to get an approximate solution of the $CP[\boldsymbol{f}]$. We will control the distance from the trajectory in such a way that $\|\boldsymbol{F}(\boldsymbol{z}) - t\boldsymbol{c}\|$ tends to zero as the right-hand side of the system (3) tends to zero along the half-line $C = \{t\boldsymbol{c} \in R^n : t > 0\}$. For this purpose, we will introduce a "cone-neighborhood" U of the half-line C. To develop a numerical method that traces the trajectory $\boldsymbol{F}^{-1}(C)$, we further assume in the remainder of the paper:

Condition 5.1.

- (i) The mapping $\boldsymbol{f}$ associated with the $CP[\boldsymbol{f}]$ is continuously differentiable.
- (ii) $\boldsymbol{c} = (\boldsymbol{a}, \boldsymbol{b}) \in R^{2n}_{++}$ and $\|\boldsymbol{c}\| = 1$; hence the half-line $C = \{t\boldsymbol{c} : t > 0\}$ lies in R^{2n}_{++}.
- (iii) $U \subset R^{2n}_{++} \cup \{\boldsymbol{0}\}$ is a closed convex cone whose interior int U contains the half-line C.
- (iv) We know a point $\boldsymbol{z}^1 = (\boldsymbol{x}^1, \boldsymbol{y}^1)$ such that $\boldsymbol{F}(\boldsymbol{z}^1) \in U$ in advance.

It is always possible to choose $\boldsymbol{c} = (\boldsymbol{a}, \boldsymbol{b})$, U and $(\boldsymbol{x}^1, \boldsymbol{y}^1)$ satisfying (ii), (iii) and (iv) of Condition 5.1. For example, choose $\boldsymbol{x}^1 > \boldsymbol{0}$, $\boldsymbol{y}^1 > \boldsymbol{0}$ and $\boldsymbol{b}' > \boldsymbol{0}$ such that $\boldsymbol{b}' = \boldsymbol{y}^1 - \boldsymbol{f}(\boldsymbol{x}^1)$. Let $\boldsymbol{a}' = (x_1^1 y_1^1, x_2^1 y_2^1, \cdots, x_n^1 y_n^1)$. Let $\boldsymbol{c} = (\boldsymbol{a}', \boldsymbol{b}')/\|(\boldsymbol{a}', \boldsymbol{b}')\|$, and ρ be a positive number such that $c_i > \rho$ $(i = 1, 2, \cdots, 2n)$. Define $U = \{\boldsymbol{u} : \|\boldsymbol{u} - t\boldsymbol{c}\| \leq t\rho$ for some $t \geq 0\}$. Then the set of $\boldsymbol{c} = (\boldsymbol{a}, \boldsymbol{b})$, U and $(\boldsymbol{x}^1, \boldsymbol{y}^1)$ satisfies Condition 5.1.

The lemma below shows some properties of the neighborhood U of the half-line C, which will be utilized in the succeeding discussions.

Lemma 5.2.

(i) *The set $U_1 = \{\boldsymbol{u} \in U : \boldsymbol{c} \cdot \boldsymbol{u} = 1\}$ is bounded.*
(ii) *There exists a positive number τ such that $\|\boldsymbol{u} - (\boldsymbol{c} \cdot \boldsymbol{u})\boldsymbol{c}\| \leq (\boldsymbol{c} \cdot \boldsymbol{u})\tau$ for each $\boldsymbol{u} \in U$.*
(iii) *There is a positive number σ such that if $\|\boldsymbol{u} - t\boldsymbol{c}\| \leq t\sigma$ for some $t > 0$ then $\boldsymbol{u} \in \operatorname{int} U$.*

Proof: (i) One can easily see that the set $\{\boldsymbol{u} \in R^{2n}_{+} : \boldsymbol{c} \cdot \boldsymbol{u} = 1\}$, which contains the set U_1, is bounded because $\boldsymbol{c} \in R^{2n}_{++}$.
(ii) Since the set $U_1 = \{\boldsymbol{u} \in U : \boldsymbol{c} \cdot \boldsymbol{u} = 1\}$ is bounded, there is a positive number τ such that the ball $B = \{\boldsymbol{u} \in R^{2n} : \|\boldsymbol{u} - \boldsymbol{c}\| \leq \tau\}$ contains the set U_1. Let $\boldsymbol{u} \in U$. Obviously $\boldsymbol{c} \cdot \boldsymbol{u} \geq 0$ because $\boldsymbol{c}, \boldsymbol{u} \in R^{2n}_{+}$. If $\boldsymbol{c} \cdot \boldsymbol{u} > 0$ then the point $\boldsymbol{u}/(\boldsymbol{c} \cdot \boldsymbol{u})$ belongs to the set $U_1 \subset B$; hence the inequality

$$\|\boldsymbol{u} - (\boldsymbol{c} \cdot \boldsymbol{u})\boldsymbol{c}\| \leq (\boldsymbol{c} \cdot \boldsymbol{u})\tau$$

follows. Now suppose that $\boldsymbol{u} \in U$ and $\boldsymbol{c} \cdot \boldsymbol{u} = 0$. Then $\boldsymbol{u} = 0$ because $\boldsymbol{c} \in R^{2n}_{++}$ and $\boldsymbol{u} \in R^{2n}_{+}$. Hence the inequality above holds trivially.
(iii) By Condition 5.1, the point $\boldsymbol{c}$ lies in the interior $\operatorname{int} U$ of the cone U. Hence, we can find a positive number σ such that $\operatorname{int} U$ contains the ball $B' = \{\boldsymbol{u} \in R^{2n} : \|\boldsymbol{u} - \boldsymbol{c}\| \leq \sigma\}$. Suppose that $\|\boldsymbol{u} - t\boldsymbol{c}\| \leq t\sigma$ for some $t > 0$. Then $\boldsymbol{u}/t$ belongs to the ball B'. Since U is a cone, we have $\boldsymbol{u} \in \operatorname{int} U$. ∎

The set $\boldsymbol{F}^{-1}(U)$ will serve as an admissible region in which we will generate a sequence $\{\boldsymbol{z}^k \in R^{2n}_{++}\}$, to approximate the trajectory $\boldsymbol{F}^{-1}(C)$, such that $\lim_{k\to\infty} \boldsymbol{c} \cdot \boldsymbol{F}(\boldsymbol{z}^k) = 0$. Such a sequence $\{\boldsymbol{z}^k\}$ leads to complementary solutions of the $CP[\boldsymbol{f}]$ as we will see in the theorem below.

Theorem 5.3. *Suppose $\{\boldsymbol{z}^k \in \boldsymbol{F}^{-1}(U)\}$ is a sequence such that $\lim_{k\to\infty}$ $\boldsymbol{c} \cdot \boldsymbol{F}(\boldsymbol{z}^k) = 0$. Then the sequence $\{\boldsymbol{z}^k\}$ is bounded and any limiting point of the sequence is a complementary solution of the $CP[\boldsymbol{f}]$.*

Proof: Since $\boldsymbol{F}(\boldsymbol{z}^k) \in U$ $(k = 1, 2, \cdots)$ holds from the assumption, we see by Lemma 5.2 that

$$\|\boldsymbol{F}(\boldsymbol{z}^k) - (\boldsymbol{c} \cdot \boldsymbol{F}(\boldsymbol{z}^k))\boldsymbol{c}\| \leq (\boldsymbol{c} \cdot \boldsymbol{F}(\boldsymbol{z}^k))\tau \quad (k = 1, 2, \cdots).$$

Hence $\lim_{k\to\infty} \boldsymbol{F}(\boldsymbol{z}^k) = \boldsymbol{0}$. Furthermore, the sequence $\{\boldsymbol{F}(\boldsymbol{z}^k)\}$ is bounded. By Remark 1.6, so is the sequence $\{\boldsymbol{z}^k\}$. Therefore we see from the continuity of the mapping $\boldsymbol{F}$ that $\boldsymbol{F}(\bar{\boldsymbol{z}}) = 0$ for any limiting point $\bar{\boldsymbol{z}}$ of the sequence $\{\boldsymbol{z}^k\}$. ∎

Assuming that we are at some point $\bar{\boldsymbol{z}} \in \boldsymbol{F}^{-1}(U) \cap R^{2n}_{++}$, i.e., $\bar{\boldsymbol{z}} \in \boldsymbol{F}^{-1}(U - \{\boldsymbol{0}\})$, we show how to generate a new point $\boldsymbol{z} \in \boldsymbol{F}^{-1}(U)$ such that $\boldsymbol{c} \cdot \boldsymbol{F}(\boldsymbol{z}) < \boldsymbol{c} \cdot \boldsymbol{F}(\bar{\boldsymbol{z}})$. This process corresponds to one iteration of the algorithm described below. Let $\bar{\beta} \in (0, 1)$ and $\phi > 0$ be fixed such that

$$(1 + \phi)\bar{\beta} < 1. \tag{7}$$

Let $\beta \in (0, \bar{\beta}]$, and $\bar{t} = \boldsymbol{c} \cdot \boldsymbol{F}(\bar{\boldsymbol{z}}) > 0$. We apply a Newton iteration with a step length $\theta \in (0, 1]$ to the system of equations: $\boldsymbol{F}(\boldsymbol{z}) = \beta\bar{t}\boldsymbol{c}$, at the point $\bar{\boldsymbol{z}}$. That is, we solve the *Newton equation*, the system of linear equations in the variable vector $\boldsymbol{\Delta z}$

$$\boldsymbol{DF}(\bar{\boldsymbol{z}})\boldsymbol{\Delta z} = \boldsymbol{F}(\bar{\boldsymbol{z}}) - \beta\bar{t}\boldsymbol{c}. \tag{8}$$

Here $\boldsymbol{DF}(\bar{\boldsymbol{z}})$ denotes the Jacobian matrix of the mapping $\boldsymbol{F}$ at $\bar{\boldsymbol{z}}$. We call $\boldsymbol{\Delta z}$ the *Newton direction*. The step length θ will be determined later by an inexact line search such that

$$\boldsymbol{c} \cdot \boldsymbol{F}(\bar{\boldsymbol{z}} - \theta\boldsymbol{\Delta z}) \leq ((1 - \theta) + \theta(1 + \phi)\beta)\bar{t}.$$

Thus we define a new point $\boldsymbol{z} \in R^{2n}$ by $\boldsymbol{z} = \bar{\boldsymbol{z}} - \theta\boldsymbol{\Delta z}$. The lemma below ensures that the $2n \times 2n$ coefficient matrix on the left hand side of the system (8) is nonsingular whenever $(\boldsymbol{x}, \boldsymbol{y}) \in R^{2n}_{++}$. Hence the system (8) consistently and uniquely determines the Newton direction $\boldsymbol{\Delta z} = \boldsymbol{DF}(\bar{\boldsymbol{z}})^{-1}(\boldsymbol{F}(\bar{\boldsymbol{z}}) - \beta\bar{t}\boldsymbol{c})$.

Lemma 5.4.

(i) *The Jacobian matrix $\boldsymbol{Df}(\boldsymbol{x})$ is a P_0-matrix at every $\boldsymbol{x} \in R^n$, i.e., for every nonzero $\boldsymbol{u} \in R^n$, there is an index i such that*

$$u_i \neq 0 \quad \textit{and} \quad u_i[\boldsymbol{Df}(\boldsymbol{x})\boldsymbol{u}]_i \geq 0.$$

(ii) *The Jacobian matrix $\boldsymbol{DF}(\boldsymbol{z})$ is nonsingular at every $\boldsymbol{z} = (\boldsymbol{x}, \boldsymbol{y}) \in R^{2n}_{++}$.*

Proof: (i) Let $\boldsymbol{x} \in R^n$ and $\boldsymbol{0} \neq \boldsymbol{u} \in R^n$. We consider a sequence $\{\boldsymbol{x} + (1/k)\boldsymbol{u} : k = 1, 2, \cdots\}$. For every $k = 1, 2, \cdots$, there is an index i such that

$$\frac{1}{k}u_i \neq 0 \quad \text{and} \quad \frac{1}{k}u_i(f_i(\boldsymbol{x} + \frac{1}{k}\boldsymbol{u}) - f_i(\boldsymbol{x})) \geq 0.$$

Since the index set $\{1, 2, \cdots, n\}$ is finite, we can find an index i such that the relation above holds for this i and infinitely many k's. For such i and k, we have

$$u_i \neq 0 \ \text{ and } \ u_i[\boldsymbol{Df}(\boldsymbol{x})\boldsymbol{u}]_i + o(\frac{1}{k})/\frac{1}{k} \geq 0.$$

Here $o(h)/h \to 0$ as $h \to 0$. Taking the limit as $k \to \infty$, we obtain

$$u_i \neq 0 \ \text{ and } \ u_i[\boldsymbol{Df}(\boldsymbol{x})\boldsymbol{u}]_i \geq 0.$$

(ii) Let $\boldsymbol{z} = (\boldsymbol{x}, \boldsymbol{y}) \in R^{2n}_{++}$. The Jacobian matrix $\boldsymbol{DF}(\boldsymbol{z})$ is written as

$$\boldsymbol{DF}(\boldsymbol{z}) = \begin{pmatrix} \boldsymbol{Y} & \boldsymbol{X} \\ -\boldsymbol{Df}(\boldsymbol{x}) & \boldsymbol{I} \end{pmatrix}.$$

where $\boldsymbol{X} = \operatorname{diag} \boldsymbol{x}$, $\boldsymbol{Y} = \operatorname{diag} \boldsymbol{y}$, and $\boldsymbol{I}$ stands for the $n \times n$ identity matrix. To see that the matrix $\boldsymbol{DF}(\boldsymbol{z})$ is nonsingular, assume on the contrary that

$$\boldsymbol{DF}(\boldsymbol{z}) \begin{pmatrix} \boldsymbol{u} \\ \boldsymbol{v} \end{pmatrix} = \boldsymbol{0}$$

or

$$\boldsymbol{Y}\boldsymbol{u} + \boldsymbol{X}\boldsymbol{v} = \boldsymbol{0} \ \text{ and } \ -\boldsymbol{Df}(\boldsymbol{x})\boldsymbol{u} + \boldsymbol{v} = \boldsymbol{0}$$

for some nonzero $(\boldsymbol{u}, \boldsymbol{v}) \in R^{2n}$. It follows that

$$\boldsymbol{u} \neq \boldsymbol{0} \ \text{ and } \ \boldsymbol{Df}(\boldsymbol{x})\boldsymbol{u} = -\boldsymbol{X}^{-1}\boldsymbol{Y}\boldsymbol{u}.$$

Hence

$$\boldsymbol{u} \neq \boldsymbol{0} \ \text{ and } \ u_i[\boldsymbol{Df}(\boldsymbol{x})\boldsymbol{u}]_i = -\frac{y_i u_i^2}{x_i} \quad (i = 1, 2, \cdots, n).$$

This contradicts (i). ∎

Remark 5.5. From (ii) of Lemma 4.3 and (ii) of Lemma 5.4, we see that $\boldsymbol{F}$ maps R^{2n}_{++} onto $R^n_{++} \times B_{++}[\boldsymbol{f}]$ diffeomorphically.

Recall that $\bar{\beta} \in (0, 1)$ and $\phi > 0$ are constants satisfying (7), and that $\boldsymbol{\Delta z}$ is a unique solution of the Newton equation (8) at $\bar{\boldsymbol{z}} \in \boldsymbol{F}^{-1}(U)$ with the parameter $\beta \in (0, \bar{\beta}]$. Assume for the time being that $\boldsymbol{z} = \bar{\boldsymbol{z}} - \theta\boldsymbol{\Delta z} \in \boldsymbol{F}^{-1}(U)$ for every sufficiently small nonnegative θ. Ideally, we want to choose the step length $\hat{\theta}$ such that

$$\begin{aligned} \boldsymbol{F}(\hat{\boldsymbol{z}}) &= \boldsymbol{F}(\bar{\boldsymbol{z}} - \hat{\theta}\boldsymbol{\Delta z}) \in U, \\ \boldsymbol{c} \cdot \boldsymbol{F}(\hat{\boldsymbol{z}}) &= \min \{\boldsymbol{c} \cdot \boldsymbol{F}(\bar{\boldsymbol{z}} - \theta\boldsymbol{\Delta z}) : \theta \in [0, 1]\}. \end{aligned}$$

However, the computation of the exact value of the ideal step length $\hat{\theta}$ is generally impossible in a finite number of steps. In the algorithm presented below, we will use an inexact line search: Find the smallest nonnegative integer $\bar{\ell}$ such that

$$
\begin{aligned}
\boldsymbol{F}(\bar{\boldsymbol{z}} - \chi^{\bar{\ell}} \boldsymbol{\Delta z}) &\in U, && (9)\\
\boldsymbol{c} \cdot \boldsymbol{F}(\bar{\boldsymbol{z}} - \chi^{\bar{\ell}} \boldsymbol{\Delta z}) &\leq \left((1 - \chi^{\bar{\ell}}) + \chi^{\bar{\ell}}(1 + \phi)\beta\right) \bar{t}. && (10)
\end{aligned}
$$

Here $\chi \in (0, 1)$ denotes a constant and $\chi^{\bar{\ell}}$ stands for the $\bar{\ell}$'th power of χ.

Now we are ready to describe the algorithm.

ALG1$[U, \beta, \chi, \phi]$.

Step 0. Let $t^1 = \boldsymbol{c} \cdot \boldsymbol{F}(\boldsymbol{z}^1)$ and $k = 1$.
Step 1. Let $\bar{\boldsymbol{z}} = \boldsymbol{z}^k$ and $\bar{t} = t^k$.
Step 2. Compute the direction $\boldsymbol{\Delta z}$ by solving the Newton equation (8).
Step 3. Let $\bar{\ell}$ be the smallest nonnegative integer satisfying (9) and (10). Define

$$\boldsymbol{z}^{k+1} = \bar{\boldsymbol{z}} - \chi^{\bar{\ell}} \boldsymbol{\Delta z} \quad \text{and} \quad t^{k+1} = \boldsymbol{c} \cdot \boldsymbol{F}(\boldsymbol{z}^{k+1}).$$

Step 4. Replace k by $k + 1$. Go to Step 1.

If it happens that $t^k = 0$ for some k in the algorithm above, then $\boldsymbol{c} \cdot \boldsymbol{F}(\boldsymbol{z}^k) = 0$; hence $\boldsymbol{F}(\boldsymbol{z}^k) = 0$. In this case we may stop the algorithm because we have obtained $\boldsymbol{z}^k$ as a complementary solution of the $CP[\boldsymbol{f}]$. So it is implicitly assumed in the algorithm above that $t^k > 0$ for every k.

6. Global and Monotone Convergence

Let $\{(\boldsymbol{z}^k, t^k)\}$ be a sequence generated by the $ALG1[U, \beta, \chi, \phi]$. We will show in Theorem 6.2 that $\lim_{k\to\infty} t^k = 0$; hence, by Theorem 5.3, the sequence $\{\boldsymbol{z}^k\}$ is bounded and any limiting point of the sequence is a complementary solution of the $CP[\boldsymbol{f}]$. For this purpose we prove the lemma below:

Lemma 6.1. *Suppose that $\bar{\boldsymbol{z}} \in \boldsymbol{F}^{-1}(U)$ and $\bar{t} = \boldsymbol{c} \cdot \boldsymbol{F}(\bar{\boldsymbol{z}}) > 0$. Let $\bar{\beta} \in (0, 1)$ and $\phi > 0$ be constants satisfying (7), and let $\boldsymbol{\Delta z}$ be the solution of the Newton equation (8) at $\bar{\boldsymbol{z}}$ with the parameter $\beta \in (0, \bar{\beta}]$.*

(i) *Define*

$$
\begin{aligned}
\theta^* &= \max\{\theta \in [0, 1] \;:\; \bar{\boldsymbol{z}} - \theta \boldsymbol{\Delta z} \in R^{2n}_+\}, && (11)\\
\boldsymbol{e}(\theta) &= \boldsymbol{F}(\bar{\boldsymbol{z}} - \theta \boldsymbol{\Delta z}) - \boldsymbol{F}(\bar{\boldsymbol{z}}) + \theta \boldsymbol{DF}(\bar{\boldsymbol{z}}) \boldsymbol{\Delta z} \quad \textit{for every } \theta \in [0, \theta^*]. && (12)
\end{aligned}
$$

Then

$$
\begin{aligned}
\lim_{\theta \to 0} \|\boldsymbol{e}(\theta)\|/\theta &= 0, && (13)\\
\boldsymbol{F}(\bar{\boldsymbol{z}} - \theta \boldsymbol{\Delta z}) &= (1-\theta)\boldsymbol{F}(\bar{\boldsymbol{z}}) + \theta(\beta \bar{t}\boldsymbol{c} + \boldsymbol{e}(\theta)/\theta) \quad \textit{for every } \theta \in (0, \theta^*], && (14)\\
\boldsymbol{c} \cdot \boldsymbol{F}(\bar{\boldsymbol{z}} - \theta \boldsymbol{\Delta z}) &\leq (1-\theta)\bar{t} + \theta(\beta \bar{t} + \|\boldsymbol{e}(\theta)\|/\theta) \quad \textit{for every } \theta \in (0, \theta^*]. && (15)
\end{aligned}
$$

(ii) *Define*

$$\bar{\theta} = \sup\{\theta' \in (0, \theta^*] \ : \ \|\boldsymbol{e}(\theta)\|/\theta < [\min\{\sigma, \ \phi\}]\beta \bar{t} \ \ \textit{for every } \theta \in (0, \theta']\}. \tag{16}$$

Then

$$0 < \bar{\theta} \leq \theta^* \leq 1 \tag{17}$$

and

$$
\begin{aligned}
\|\boldsymbol{e}(\theta)\|/\theta &< [\min\{\sigma, \ \phi\}]\beta \bar{t}, && (18)\\
\boldsymbol{F}(\bar{\boldsymbol{z}} - \theta \boldsymbol{\Delta z}) &\in \operatorname{int} U, && (19)\\
\boldsymbol{c} \cdot \boldsymbol{F}(\bar{\boldsymbol{z}} - \theta \boldsymbol{\Delta z}) &< ((1-\theta) + \theta(1+\phi)\beta)\, \bar{t} && (20)
\end{aligned}
$$

for every $\theta \in (0, \bar{\theta})$.

Proof: (i) It should be noticed that $\theta^* \in (0, 1]$. The relation (13) follows directly from the continuous differentiability of the mapping $\boldsymbol{F}$. By the definition,

$$\boldsymbol{F}(\bar{\boldsymbol{z}} - \theta \boldsymbol{\Delta z}) = \boldsymbol{F}(\bar{\boldsymbol{z}}) - \theta \boldsymbol{DF}(\bar{\boldsymbol{z}})\boldsymbol{\Delta z} + \boldsymbol{e}(\theta) \ \ \text{for every } \theta \in [0, \theta^*].$$

Since $\boldsymbol{\Delta z}$ is the solution of the Newton equation (8), we also have

$$\boldsymbol{F}(\bar{\boldsymbol{z}}) - \theta \boldsymbol{DF}(\bar{\boldsymbol{z}})\boldsymbol{\Delta z} = (1-\theta)\boldsymbol{F}(\bar{\boldsymbol{z}}) + \theta \beta \bar{t}\boldsymbol{c} \ \ \text{for every } \theta \in [0, \theta^*].$$

Hence the equality (14) follows from these two equalities. Taking the inner product of each side of (14) and the vector $\boldsymbol{c}$, we have that

$$
\begin{aligned}
\boldsymbol{c} \cdot \boldsymbol{F}(\bar{\boldsymbol{z}} - \theta \boldsymbol{\Delta z}) &= (1-\theta)\boldsymbol{c} \cdot \boldsymbol{F}(\bar{\boldsymbol{z}}) + \theta(\beta \bar{t}\|\boldsymbol{c}\|^2 + \boldsymbol{c} \cdot \boldsymbol{e}(\theta)/\theta)\\
&\leq (1-\theta)\bar{t} + \theta(\beta \bar{t} + \|\boldsymbol{e}(\theta)\|/\theta)
\end{aligned}
$$

for every $\theta \in (0, \theta^*]$. Thus we have shown the inequality (15).
(ii) The inequality (17) follows from the definitions (11), (16) of θ^*, $\bar{\theta}$ and the relation (13). The inequality (18) is obvious by the definition (16) of $\bar{\theta}$, too. Let $\theta \in (0, \bar{\theta})$. By Lemma 5.2, the point $(\beta \bar{t}\boldsymbol{c} + \boldsymbol{e}(\theta)/\theta)$ lies in int U. Since the point $\boldsymbol{F}(\bar{\boldsymbol{z}} - \theta \boldsymbol{\Delta z})$ is a convex combination of the point $\boldsymbol{F}(\bar{\boldsymbol{z}})$ in the convex set U and the point $(\beta \bar{t}\boldsymbol{c} + \boldsymbol{e}(\theta)/\theta)$ in int U, it lies in int U. Thus we have shown (19). The inequality (20) follows from (15) and (18). ∎

Lemma 6.1 guarantees that we can consistently find the smallest nonnegative integer $\bar{\ell}$ satisfying (9) and (10) at Step 3 of $ALG1[U, \beta, \chi, \phi]$. We are now ready to prove the global and monotone convergence property of $ALG1[U, \beta, \chi, \phi]$.

Theorem 6.2. *Let $\bar{\beta} \in (0,1)$ and $\phi > 0$ be constants satisfying (7). Suppose that $\beta \in (0, \bar{\beta}]$ and $\chi \in (0,1)$. Let $\{(\boldsymbol{z}^k, t^k)\}$ be a sequence generated by the $ALG1[U, \beta, \chi, \phi]$.*

(i) *The sequence $\{t^k\}$ is monotone decreasing and converges to zero as $k \to \infty$.*
(ii) *The sequence $\{\boldsymbol{z}^k\}$ is bounded and its limiting points are complementary solutions of the $CP[\boldsymbol{f}]$.*

Proof: In view of Theorem 5.3, it suffices to prove the assertion (i). By applying Lemma 6.1 at each $\bar{\boldsymbol{z}} = \boldsymbol{z}^k$, we see that $t^k > t^{k+1}$ $(k = 1, 2, \cdots)$. Hence the sequence $\{t^k\}$ is monotone decreasing. Since each t^k is nonnegative, there exists a nonnegative number $\tilde{t}$ to which the sequence converges. If $\tilde{t} = 0$, we obtain the desired result. Assume on the contrary that $\tilde{t} > 0$. Define the compact subset

$$V = \{\boldsymbol{u} \in U : \tilde{t} \leq \boldsymbol{c} \cdot \boldsymbol{u} \leq t^1\}$$

of R^{2n}_{++} (see Lemma 5.2). Then we see by (ii) of Lemma 4.3 that the set $\boldsymbol{F}^{-1}(V)$ which contains the sequence $\{\boldsymbol{z}^k\}$ is a compact subset of R^{2n}_{++} since $V \subset R^{2n}_{++} \subset R^n_{++} \times B_{++}[\boldsymbol{f}]$. Taking a subsequence if necessary, we may assume that the sequence $\{\boldsymbol{z}^k\}$ converges to some $\tilde{\boldsymbol{z}} \in \boldsymbol{F}^{-1}(V)$. Then it is easily seen that $\tilde{t} = \boldsymbol{c} \cdot \boldsymbol{F}(\tilde{\boldsymbol{z}})$. Now, applying Lemma 6.1 to the point $\tilde{\boldsymbol{z}}$, we can find a positive number $\tilde{\theta}$ such that for every $\theta \in (0, \tilde{\theta})$,

$$\begin{array}{rcl} \boldsymbol{F}(\tilde{\boldsymbol{z}} - \theta \widetilde{\boldsymbol{\Delta z}}) & \in & \mathrm{int}\, U, \\ \boldsymbol{c} \cdot \boldsymbol{F}(\tilde{\boldsymbol{z}} - \theta \widetilde{\boldsymbol{\Delta z}}) & < & ((1-\theta) + \theta(1+\phi)\beta)\, \tilde{t}\ . \end{array}$$

Here $\widetilde{\boldsymbol{\Delta z}}$ denotes the Newton direction determined by the equation (8) with $\bar{\boldsymbol{z}} = \tilde{\boldsymbol{z}}$ and $\bar{t} = \tilde{t}$. On the other hand, the Jacobian matrix $\boldsymbol{DF}(\boldsymbol{z})$ is nonsingular and continuous at $\boldsymbol{z} = \tilde{\boldsymbol{z}}$ (see (ii) of Lemma 5.4). This implies that the Newton direction generated at the k'th iteration, $\boldsymbol{\Delta z}^k$, converges to $\widetilde{\boldsymbol{\Delta z}}$. Therefore, for a nonnegative integer ℓ such that $\chi^\ell \in (0,\ \tilde{\theta})$, we have

$$\begin{array}{rcl} \boldsymbol{F}(\boldsymbol{z}^k - \chi^\ell \boldsymbol{\Delta z}^k) & \in & \mathrm{int}\, U \quad \text{and} \\ \boldsymbol{c} \cdot \boldsymbol{F}(\boldsymbol{z}^k - \chi^\ell \boldsymbol{\Delta z}^k) & < & \Big((1-\chi^\ell) + \chi^\ell(1+\phi)\beta\Big)\, t^k \end{array}$$

for every sufficiently large k. Let $\bar{\ell}^k$ be the nonnegative integer determined at Step 3 of the k'th iteration in $ALG1[U, \beta, \chi, \phi]$. Then, for every sufficiently large k, we see that $\bar{\ell}^k \leq \ell$; hence

$$\begin{array}{rcl} t^{k+1} & \leq & \Big((1-\chi^{\bar{\ell}^k}) + \chi^{\bar{\ell}^k}(1+\phi)\beta\Big)\, t^k \\ & \leq & \Big((1-\chi^\ell) + \chi^\ell(1+\phi)\beta\Big)\, t^k. \end{array}$$

This contradicts the fact that the sequence $\{t^k\}$ converges to $\tilde{t}$. ∎

7. Local Convergence

We will assume the condition below in addition to Conditions 1.5 and 5.1 throughout this section, which is divided into two subsections, 7.1 and 7.2. Subsection 7.1 is devoted to a locally linear convergence property of $ALG1[U, \beta, \chi, \phi]$. In Subsection 7.2 we will modify $ALG1[U, \beta, \chi, \phi]$ to get a locally quadratic convergence.

Condition 7.1.

(i) At each complementary solution $\boldsymbol{z} = (\boldsymbol{x}, \boldsymbol{y})$ of the $CP[\boldsymbol{f}]$, the set of the columns $\boldsymbol{I}_i$ $(i \in I_+(\boldsymbol{y}))$ and $[\boldsymbol{Df}(\boldsymbol{x})]_j$ $(j \in I_+(\boldsymbol{x}))$ forms a basis of R^n. Here $\boldsymbol{I}_i$ denotes the i'th column of the $n \times n$ identity matrix $\boldsymbol{I}$, $[\boldsymbol{Df}(\boldsymbol{x})]_j$ the j'th column of the $n \times n$ Jacobian matrix $\boldsymbol{Df}(\boldsymbol{x})$ of the mapping $\boldsymbol{f}$, $I_+(\boldsymbol{y}) = \{i : y_i > 0\}$, and $I_+(\boldsymbol{x}) = \{j : x_j > 0\}$.

(ii) The Jacobian matrix $\boldsymbol{Df}(\boldsymbol{x})$ of the mapping $\boldsymbol{f}$ is Lipschitz continuous on each bounded subset $E \subset R^n_+$, i.e., there is a positive constant λ such that

$$\|\boldsymbol{Df}(\boldsymbol{x}^2) - \boldsymbol{Df}(\boldsymbol{x}^1)\| \leq \lambda \|\boldsymbol{x}^2 - \boldsymbol{x}^1\| \quad \text{for every } \boldsymbol{x}^1,\ \boldsymbol{x}^2 \in E,$$

where $\|\boldsymbol{A}\|$ denotes the matrix norm $\max\{\|\boldsymbol{Aw}\| : \boldsymbol{w} \in R^n,\ \|\boldsymbol{w}\| = 1\}$ for every $n \times n$ matrix $\boldsymbol{A}$.

We note that (i) of Condition 7.1 implies the strict complementarity, i.e., $x_i = 0$ if and only if $y_i > 0$ $(i = 1, 2, \cdots, n)$. By using the well-known implicit function theorem, we can also derive the local uniqueness of each solution of the $CP[\boldsymbol{f}]$ from (i) of Condition 7.1. Furthermore we will see in Lemma 7.3 below that the $CP[\boldsymbol{f}]$ has a unique solution.

7.1 Locally Linear Convergence

Now we state the locally linear convergence of $ALG1[U, \beta, \chi, \phi]$.

Theorem 7.2. *Let $\bar{\beta} \in (0, 1)$ and $\phi > 0$ be constants satisfying (7). Suppose that $\beta \in (0, \bar{\beta}]$ and $\chi \in (0, 1)$. Let $\{(\boldsymbol{z}^k, t^k)\}$ be a sequence generated by the $ALG1[U, \beta, \chi, \phi]$. Then there is a positive number K such that*

$$t^{k+1} \leq (1 + \phi)\beta t^k \quad \textit{for every } k \geq K.$$

We will prove a series of lemmas which leads us to Theorem 7.2.

Lemma 7.3.

(i) *The Jacobian matrix* $\boldsymbol{DF}(\boldsymbol{z})$ *of the mapping* $\boldsymbol{F}$ *is nonsingular at every* $\boldsymbol{z} = (\boldsymbol{x}, \boldsymbol{y}) \in R^{2n}_{++} \cup \boldsymbol{F}^{-1}(\boldsymbol{0})$.

(ii) *The* $CP[\boldsymbol{f}]$ *has a unique solution.*

(iii) *There are positive constants* ξ *and* η *such that*

$$\begin{aligned} \|\boldsymbol{DF}(\boldsymbol{z})\| &\leq \xi, & (21)\\ \|\boldsymbol{DF}(\boldsymbol{z})^{-1}\| &\leq \eta & (22) \end{aligned}$$

for every $\boldsymbol{z} \in \boldsymbol{F}^{-1}(U(t^1))$ *and*

$$\begin{aligned} \|\boldsymbol{F}(\boldsymbol{z}^2) - \boldsymbol{F}(\boldsymbol{z}^1)\| &\leq \xi\|\boldsymbol{z}^2 - \boldsymbol{z}^1\|, & (23)\\ \|\boldsymbol{z}^2 - \boldsymbol{z}^1\| &\leq \eta\|\boldsymbol{F}(\boldsymbol{z}^2) - \boldsymbol{F}(\boldsymbol{z}^1)\| & (24) \end{aligned}$$

for every $\boldsymbol{z}^1, \boldsymbol{z}^2 \in \boldsymbol{F}^{-1}(U(t^1))$.

(iv) *There is a positive constant* κ *such that*

$$\|\boldsymbol{F}(\boldsymbol{z}^2) - \boldsymbol{F}(\boldsymbol{z}^1) - \boldsymbol{DF}(\boldsymbol{z}^1)(\boldsymbol{z}^2 - \boldsymbol{z}^1)\| \leq \kappa\|\boldsymbol{z}^2 - \boldsymbol{z}^1\|^2$$

for every $\boldsymbol{z}^1, \boldsymbol{z}^2 \in \boldsymbol{F}^{-1}(U(t^1))$.

Here $U(t^1) = \{\boldsymbol{u} \in U : \boldsymbol{c} \cdot \boldsymbol{u} \leq t^1\}$.

Proof: (i) Recall the definition (1) of the mapping $\boldsymbol{F} : R^{2n}_{+} \rightarrow R^{n}_{+} \times R^{n}$. If $\boldsymbol{z} \in R^{2n}_{++} \cup \boldsymbol{F}^{-1}(\boldsymbol{0})$ then we have either

$$\boldsymbol{z} = (\boldsymbol{x}, \boldsymbol{y}) \in R^{2n}_{++}$$

or

$$\boldsymbol{F}(\boldsymbol{z}) = \boldsymbol{0}, \ \boldsymbol{z} = (\boldsymbol{x}, \boldsymbol{y}) \in R^{2n}_{+}.$$

Note that $\boldsymbol{z}$ is a complementary solution of the $CP[\boldsymbol{f}]$ in the latter case. We have shown in Lemma 5.4 that the Jacobian matrix $\boldsymbol{DF}(\boldsymbol{z})$ is nonsingular at every $\boldsymbol{z} \in R^{2n}_{++}$. Now suppose that $\boldsymbol{F}(\boldsymbol{z}) = \boldsymbol{0}$ and $\boldsymbol{z} \in R^{2n}_{+}$. By (i) of Condition 7.1, we can easily verify that if $\boldsymbol{DF}(\boldsymbol{z})\boldsymbol{w} = \boldsymbol{0}$ for some $\boldsymbol{w} \in \boldsymbol{R}^{2n}$ then $\boldsymbol{w} = \boldsymbol{0}$; hence $\boldsymbol{DF}(\boldsymbol{z})$ is nonsingular.
(ii) By Theorem 4.4, we know that the $CP[\boldsymbol{f}]$ has a complementary solution. On the other hand, by applying the implicit function theorem (see, for example, Ortega and Rheinboldt [22]) to the system (3) at each complementary solution $\bar{\boldsymbol{z}}$ of the $CP[\boldsymbol{f}]$ and $\bar{t} = 0$, we see that the unique trajectory $\boldsymbol{F}^{-1}(C)$, whose existence is ensured by Theorem 4.4, converges to $\bar{\boldsymbol{z}}$ as $t \rightarrow 0$. Hence the solution of the $CP[\boldsymbol{f}]$ must be unique.
(iii) Let $\boldsymbol{z}^*$ be the unique solution of the $CP[\boldsymbol{f}]$. Since

$$U(t^1) \subset U \subset R^{2n}_{++} \cup \{\boldsymbol{0}\} \subset (R^{n}_{++} \times B_{++}[\boldsymbol{f}]) \cup \{\boldsymbol{0}\} \ ,$$

we know $\boldsymbol{F}^{-1}(U(t^1)) \subset R^{2n}_{++} \cup \{\boldsymbol{z}^*\}$. Noting (ii) of Lemma 4.3 and extending slightly the argument in (ii) above, we can show that $\boldsymbol{F}$ is a homeomorphism between $\boldsymbol{F}^{-1}(U(t^1))$ and $U(t^1)$. On the other hand, the set $U(t^1)$ is compact by Lemma 5.2. Therefore the set $\boldsymbol{F}^{-1}(U(t^1))$ is compact, too. Let W be the convex hull of the compact set $\boldsymbol{F}^{-1}(U(t^1))$. Then W is also compact and $W \subset R^{2n}_{++} \cup \{\boldsymbol{z}^*\}$. We have seen by (i) above that the Jacobian matrix $\boldsymbol{DF}(\boldsymbol{z})$ is nonsingular at every $\boldsymbol{z}$ in the set $R^{2n}_{++} \cup \{\boldsymbol{z}^*\}$. Thus, by the continuity of the Jacobian matrix $\boldsymbol{DF}(\boldsymbol{z})$, there exist positive numbers ξ and η such that (21) Holds for every $\boldsymbol{z} \in W$ and that (22) holds for every $\boldsymbol{z} \in \boldsymbol{F}^{-1}(U(t^1))$. Since the sets W and $U(t^1)$ are convex, the inequalities (23) and (24) follow from (21), (22) and Theorem 3.2.3 of [22] .
(iv) It follows from (ii) of Condition 7.1 that the Jacobian matrix $\boldsymbol{DF}(\boldsymbol{z})$ is Lipschitz continuous on the bounded convex set W, i.e., there is a positive number κ such that

$$\|\boldsymbol{DF}(\boldsymbol{z}^2) - \boldsymbol{DF}(\boldsymbol{z}^1)\| \leq 2\kappa\|\boldsymbol{z}^2 - \boldsymbol{z}^1\| \text{ for every } \boldsymbol{z}^1,\ \boldsymbol{z}^2 \in W.$$

The assertion (iv) follows from Theorem 3.2.12 of [22]. ∎

Lemma 7.4. *Let $\bar{\boldsymbol{z}} \in \boldsymbol{F}^{-1}(U(t^1))$, $\bar{t} = \boldsymbol{c} \cdot \boldsymbol{F}(\bar{\boldsymbol{z}}) > 0$, and let $\boldsymbol{\Delta z}$ be the solution of the Newton equation (8) at $\bar{\boldsymbol{z}}$ with the parameter $\beta \in (0, \bar{\beta}]$. Define $\boldsymbol{e}(\theta)$ and $\bar{\theta}$ by (12) and (16), respectively. Then*

$$\|\boldsymbol{e}(\theta)\| \leq \kappa\left(\eta(\tau + 1 - \beta)\bar{t}\theta\right)^2 \quad \textit{for every } \theta \in [0, \bar{\theta}]. \tag{25}$$

Proof: Let $\theta \in [0, \bar{\theta})$. Since $\boldsymbol{\Delta z}$ is the solution of the Newton equation (8), we have

$$\begin{aligned}
\|\boldsymbol{\Delta z}\| &= \|\boldsymbol{DF}(\bar{\boldsymbol{z}})^{-1}(\boldsymbol{F}(\bar{\boldsymbol{z}}) - \beta\bar{t}\boldsymbol{c})\| \\
&\leq \|\boldsymbol{DF}(\bar{\boldsymbol{z}})^{-1}\|\left(\|\boldsymbol{F}(\bar{\boldsymbol{z}}) - \bar{t}\boldsymbol{c}\| + (1-\beta)\bar{t}\|\boldsymbol{c}\|\right) \\
&\leq \eta\left(\|\boldsymbol{F}(\bar{\boldsymbol{z}}) - \bar{t}\boldsymbol{c}\| + (1-\beta)\bar{t}\right) \quad \text{(by Lemma 7.3 and } \|\boldsymbol{c}\| = 1\text{)} \\
&\leq \eta\left(\tau\bar{t} + (1-\beta)\bar{t}\right) \quad \text{(by Lemma 5.2).}
\end{aligned}$$

Thus we have seen that

$$\|\boldsymbol{\Delta z}\| \leq \eta(\tau + 1 - \beta)\bar{t}. \tag{26}$$

On the other hand, by the relation (19), (20) in Lemma 6.1 and the continuity, we see for every $\theta \in [0, \bar{\theta}]$ that

$$\begin{aligned}
\boldsymbol{F}(\bar{\boldsymbol{z}} - \theta\boldsymbol{\Delta z}) &\in U, \\
\boldsymbol{c} \cdot \boldsymbol{F}(\bar{\boldsymbol{z}} - \theta\boldsymbol{\Delta z}) &\leq ((1-\theta) + \theta(1+\phi)\beta)\,\bar{t} \ \leq \bar{t} \ \leq t^1;
\end{aligned}$$

hence

$$\bar{\boldsymbol{z}} - \theta\boldsymbol{\Delta z} \in \boldsymbol{F}^{-1}(U(t^1)).$$

Therefore, by the definition (12) of $\boldsymbol{e}(\theta)$ and (iv) of Lemma 7.3, we have

$$\|\boldsymbol{e}(\theta)\| \leq \kappa \|\theta \boldsymbol{\Delta z}\|^2.$$

The desired inequality follows from (26) and the inequality above. ∎

For each $t > 0$ and $\beta \in (0, \bar{\beta}]$, define

$$\hat{\theta}(t, \beta) = \min \left\{ \frac{\beta \min\{\sigma,\ \phi\}}{\kappa \left(\eta(\tau + 1 - \beta)\right)^2 t}, 1 \right\}.$$

For every $\beta \in (0, \bar{\beta}]$, let

$$s(\beta) = \max\{t' \geq 0 : \hat{\theta}(t, \beta) = 1 \ \text{ for every } \ t \in (0, t']\} = \frac{\beta \min\{\sigma,\ \phi\}}{\kappa \left(\eta(\tau + 1 - \beta)\right)^2}. \tag{27}$$

Here τ and σ are the positive constants that were introduced in Lemma 5.2.

Lemma 7.5. *Let $\bar{\boldsymbol{z}} \in \boldsymbol{F}^{-1}(U(t^1))$, $\bar{t} = \boldsymbol{c} \cdot \boldsymbol{F}(\bar{\boldsymbol{z}}) > 0$, and let $\boldsymbol{\Delta z}$ be the solution of the Newton equation (8) at $\bar{\boldsymbol{z}}$ with the parameter $\beta \in (0, \bar{\beta}]$. Then*

$$\begin{aligned} \boldsymbol{F}(\bar{\boldsymbol{z}} - \theta \boldsymbol{\Delta z}) &\in U, & (28)\\ \boldsymbol{c} \cdot \boldsymbol{F}(\bar{\boldsymbol{z}} - \theta \boldsymbol{\Delta z}) &\leq ((1-\theta) + \theta(1+\phi)\beta)\bar{t} & (29) \end{aligned}$$

for every $\theta \in [0, \hat{\theta}(\bar{t}, \beta)]$.

Proof: Define $\boldsymbol{e}(\theta)$ and $\bar{\theta}$ by (12) and (16), respectively. If $\hat{\theta}(\bar{t}, \beta) \leq \bar{\theta}$ then the relations (28) and (29) follow from (19), (20) in Lemma 6.1 and the continuity. Thus it suffices to show that $\hat{\theta}(\bar{t}, \beta) \leq \bar{\theta}$. We may assume $\bar{\theta} < 1$. Suppose that

$$\|\boldsymbol{e}(\bar{\theta})\| / \bar{\theta} < [\min\{\sigma,\ \phi\}] \beta \bar{t}. \tag{30}$$

Then from the definition (16) of $\bar{\theta}$ and the continuity of $\boldsymbol{e}(\theta)$ it follows that $\bar{\theta} = \theta^*$. Just as we derived (19) from (18) in the proof of Lemma 6.1, we see from (30) that

$$\boldsymbol{F}(\bar{\boldsymbol{z}} - \theta^* \boldsymbol{\Delta z}) \in \operatorname{int} U.$$

This contradicts the definition (11) of θ^* since $\theta^* = \bar{\theta} < 1$. Thus we have shown that

$$\|\boldsymbol{e}(\bar{\theta})\| / \bar{\theta} \geq [\min\{\sigma,\ \phi\}] \beta \bar{t}.$$

By the inequality (25) in Lemma 7.4 and the inequality above, we have

$$[\min\{\sigma,\ \phi\}] \beta \bar{t} \leq \kappa \left(\eta(\tau + 1 - \beta)\bar{t}\right)^2 \bar{\theta};$$

hence $\hat{\theta}(\bar{t}, \beta) \leq \bar{\theta}$. ∎

Lemma 7.6. *Let $\bar{\boldsymbol{z}} \in \boldsymbol{F}^{-1}(U(t^1))$, $\bar{t} = \boldsymbol{c} \cdot \boldsymbol{F}(\bar{\boldsymbol{z}}) > 0$, and let $\boldsymbol{\Delta z}$ be the solution of the Newton equation (8) at $\bar{\boldsymbol{z}}$ with the parameter $\beta \in (0, \bar{\beta}]$. If $\bar{t} \leq s(\beta)$ then*

$$\begin{aligned} \boldsymbol{F}(\bar{\boldsymbol{z}} - \boldsymbol{\Delta z}) &\in U, \\ \boldsymbol{c} \cdot \boldsymbol{F}(\bar{\boldsymbol{z}} - \boldsymbol{\Delta z}) &\leq (1+\phi)\beta\bar{t}. \end{aligned}$$

Proof: By the definition (27) of $s(\beta)$ and Lemma 7.5, the relations (28) and (29) hold for every $\theta \in [0, 1]$. ∎

We are now ready to prove Theorem 7.2. We have already shown in Theorem 6.2 that the sequence $\{t^k\}$ converges to zero as $k \to \infty$. Let K be a positive integer such that $t^k \leq s(\beta)$ for every $k \geq K$. Then, by Lemma 7.6, the inequality in the theorem holds for every $k \geq K$. This completes the proof of Theorem 7.2.

7.2 A Modification of $ALG1[U, \beta, \chi, \phi]$ and Its Locally Quadratic Convergence

The integer K in Theorem 7.2 depends on the positive number β, but we can take any positive value of $\beta \in (0, \bar{\beta}]$ although K may diverge as β tends to zero. This suggests that the sequence $\{t^k\}$ converges to zero at least super-linearly if we suitably decrease the value of the parameter β as the iteration proceeds. In fact, we can modify the algorithm so that the sequence converges to zero quadratically.

Let $\bar{\boldsymbol{z}} \in \boldsymbol{F}^{-1}(U(t^1))$ and $\bar{t} = \boldsymbol{c} \cdot \boldsymbol{F}(\bar{\boldsymbol{z}}) > 0$. In the remainder of the section we denote the solution of the Newton equation (8) at $\bar{\boldsymbol{z}}$ with the parameter $\beta \in (0, \bar{\beta}]$ by $\boldsymbol{\Delta z}(\beta)$. We will be concerned with the following two sets of relations:

$$\begin{aligned} \boldsymbol{F}(\bar{\boldsymbol{z}} - \chi^{\bar{\ell}} \boldsymbol{\Delta z}(\bar{\beta})) &\in U, && (31) \\ \boldsymbol{c} \cdot \boldsymbol{F}(\bar{\boldsymbol{z}} - \chi^{\bar{\ell}} \boldsymbol{\Delta z}(\bar{\beta})) &\leq \left((1-\chi^{\bar{\ell}}) + \chi^{\bar{\ell}}(1+\phi)\bar{\beta}\right)\bar{t}, && (32) \end{aligned}$$

and

$$\begin{aligned} \boldsymbol{F}(\bar{\boldsymbol{z}} - \boldsymbol{\Delta z}(\bar{\beta}^m)) &\in U, && (33) \\ \boldsymbol{c} \cdot \boldsymbol{F}(\bar{\boldsymbol{z}} - \boldsymbol{\Delta z}(\bar{\beta}^m)) &\leq (1+\phi)\bar{\beta}^m\bar{t}. && (34) \end{aligned}$$

Here $\chi^{\bar{\ell}}$ and $\bar{\beta}^m$ represent the $\bar{\ell}$'th power of $\chi \in (0, 1)$ and the m'th power of $\bar{\beta} \in (0, 1)$, respectively. Now we are in position to state a modification of $ALG1[U, \beta, \chi, \phi]$ whose local convergence will be investigated later.

ALG2$[U, \bar{\beta}, \chi, \phi]$.

Step 0. Let $t^1 = \boldsymbol{c} \cdot \boldsymbol{F}(\boldsymbol{z}^1)$ and $k = 1$.
Step 1. Let $\bar{\boldsymbol{z}} = \boldsymbol{z}^k$ and $\bar{t} = t^k$.

Step 2. Let $\boldsymbol{\Delta z}(\bar{\beta})$ be the direction determined by the Newton equation (8) with the parameter $\beta = \bar{\beta}$. Let $\bar{\ell}$ be the smallest nonnegative integer satisfying the relations (31) and (32). If $\bar{\ell} = 0$ then go to Step 4.

Step 3. Define

$$\boldsymbol{z}^{k+1} = \bar{\boldsymbol{z}} - \chi^{\bar{\ell}} \boldsymbol{\Delta z}(\bar{\beta}).$$

Go to Step 5.

Step 4. Let $\bar{m}$ be the largest positive integer such that the relations (33) and (34) hold for all $m = 1, 2, \cdots, \bar{m}$. Define

$$\boldsymbol{z}^{k+1} = \bar{\boldsymbol{z}} - \boldsymbol{\Delta z}(\bar{\beta}^{\bar{m}}).$$

Step 5. Let $t^{k+1} = \boldsymbol{c} \cdot \boldsymbol{F}(\boldsymbol{z}^{k+1})$. Replace k by $k+1$. Go to Step 1.

Let $\{(\boldsymbol{z}^k, t^k)\}$ be a sequence generated by $ALG2[U, \bar{\beta}, \chi, \phi]$. Then, in a way similar to the proof of Theorem 6.2, we can show that the sequence $\{(\boldsymbol{z}^k, t^k)\}$ converges to $(\boldsymbol{z}^*, 0)$. Here $\boldsymbol{z}^*$ is the unique solution of the $CP[\boldsymbol{f}]$ (see (ii) of Lemma 7.3). Furthermore, $ALG2[U, \bar{\beta}, \chi, \phi]$ has a locally quadratic convergence property as we will see in Theorem 7.8 below.

Lemma 7.7. *Let $\bar{\boldsymbol{z}} \in \boldsymbol{F}^{-1}(U(t^1))$ and $\bar{t} = \boldsymbol{c} \cdot \boldsymbol{F}(\bar{\boldsymbol{z}}) > 0$. Suppose that $\bar{t} \leq s(\bar{\beta})$, where $s : (0, \bar{\beta}] \rightarrow R_{++}$ is defined by (27). Then the relations (31) and (32) hold for $\bar{\ell} = 0$, i.e., the relations (33) and (34) hold for $m = 1$. Let $\bar{m}$ be the largest positive integer such that the relations (33) and (34) hold for $m = 1, 2, \cdots, \bar{m}$. Then*

$$\boldsymbol{c} \cdot \boldsymbol{F}(\bar{\boldsymbol{z}} - \boldsymbol{\Delta z}(\bar{\beta}^{\bar{m}})) \leq \frac{\kappa(\eta(\tau+1))^2(1+\phi)}{\bar{\beta}\min\{\sigma, \phi\}} \bar{t}^2.$$

Proof: Let $\tilde{m}$ be the largest integer such that

$$\bar{t} \leq s(\bar{\beta}^m)$$

for every $m = 1, 2, \cdots, \tilde{m}$. Then $\tilde{m} \geq 1$ since $\bar{t} \leq s(\bar{\beta})$. From Lemma 7.6, we see that the relations (33) and (34) hold for $m = 1, 2, \cdots, \tilde{m}$. Hence $\tilde{m} \leq \bar{m}$. On the other hand, it follows from the definition of $\tilde{m}$ that

$$\begin{aligned} \bar{t} &> s(\bar{\beta}^{\tilde{m}+1}) \\ &= \frac{\bar{\beta}^{\tilde{m}+1} \min\{\sigma, \phi\}}{\kappa\left(\eta(\tau+1-\bar{\beta}^{\tilde{m}+1})\right)^2} \end{aligned}$$

or equivalently

$$\bar{\beta}^{\tilde{m}} < \frac{\kappa\left(\eta(\tau+1-\bar{\beta}^{\tilde{m}+1})\right)^2 \bar{t}}{\bar{\beta}\min\{\sigma, \phi\}}.$$

Since $\tilde{m} \leq \bar{m}$, we have

$$\bar{\beta}^{\bar{m}} < \frac{\kappa\,(\eta(\tau+1))^2\,\bar{t}}{\bar{\beta}\min\{\sigma,\phi\}}.$$

Therefore we obtain

$$\begin{aligned}
\boldsymbol{c}\cdot\boldsymbol{F}(\bar{\boldsymbol{z}}-\boldsymbol{\Delta z}(\bar{\beta}^{\bar{m}})) &\leq (1+\phi)\bar{\beta}^{\bar{m}}\bar{t} \\
&< \frac{\kappa\,(\eta(\tau+1))^2\,(1+\phi)}{\bar{\beta}\min\{\sigma,\phi\}}\bar{t}^2.
\end{aligned}$$

∎

Theorem 7.8. *Let $\{(\boldsymbol{z}^k,t^k)\}$ be a sequence generated by the ALG2$[U,\bar{\beta},\chi,\phi]$. Then the convergence of the sequence $\{(\boldsymbol{z}^k,t^k)\}$ to $(\boldsymbol{z}^*,0)$ is locally quadratic. More precisely,*

$$t^{k+1} \leq \frac{\kappa\,(\eta(\tau+1))^2\,(1+\phi)}{\bar{\beta}\min\{\sigma,\phi\}}(t^k)^2, \tag{35}$$

$$\|\boldsymbol{z}^{k+1}-\boldsymbol{z}^*\| \leq \frac{\kappa\xi^2\,(\eta(\tau+1))^3\,(1+\phi)}{\bar{\beta}\min\{\sigma,\phi\}}\|\boldsymbol{z}^k-\boldsymbol{z}^*\|^2 \tag{36}$$

for every sufficiently large k. Here $\boldsymbol{z}^*$ is the unique solution of the $CP[\boldsymbol{f}]$.

Proof: As we have noted above, the sequence $\{(\boldsymbol{z}^k,t^k)\}$ converges to $(\boldsymbol{z}^*,0)$ as $k\to\infty$. Let K' be a positive integer such that $t^k \leq s(\bar{\beta})$ for every $k \geq K'$. Then, the inequality (35) (for every $k\geq K'$) follows directly from Lemma 7.7. Furthermore, we have

$$\begin{aligned}
\|\boldsymbol{z}^{k+1}-\boldsymbol{z}^*\| &\leq \eta\|\boldsymbol{F}(\boldsymbol{z}^{k+1})\| \quad \text{(by (24) and } \boldsymbol{F}(\boldsymbol{z}^*)=\boldsymbol{0}\text{)} \\
&\leq \eta(\|\boldsymbol{F}(\boldsymbol{z}^{k+1})-t^{k+1}\boldsymbol{c}\|+\|t^{k+1}\boldsymbol{c}\|) \\
&\leq \eta(\tau t^{k+1}+t^{k+1}) \quad \text{(by Lemma 5.2 and } \|\boldsymbol{c}\|=1\text{)} \\
&= \eta(\tau+1)t^{k+1} \\
&\leq \frac{\kappa\,(\eta(\tau+1))^3\,(1+\phi)}{\bar{\beta}\min\{\sigma,\phi\}}(t^k)^2 \quad \text{(by (35))} \\
&\leq \frac{\kappa\,(\eta(\tau+1))^3\,(1+\phi)}{\bar{\beta}\min\{\sigma,\phi\}}\|\boldsymbol{F}(\boldsymbol{z}^k)\|^2 \quad \text{(since } t^k=\boldsymbol{c}\cdot\boldsymbol{F}(\boldsymbol{z}^k)\text{ and } \|\boldsymbol{c}\|=1\text{)} \\
&\leq \frac{\kappa\,(\eta(\tau+1))^3\,(1+\phi)}{\bar{\beta}\min\{\sigma,\phi\}}\xi^2\|\boldsymbol{z}^k-\boldsymbol{z}^*\|^2 \quad \text{(by } \boldsymbol{F}(\boldsymbol{z}^*)=\boldsymbol{0}\text{ and (23))}
\end{aligned}$$

for every $k\geq K'$. Thus we have shown (36). ∎

8. Concluding Remarks

We have formulated the complementarity problem $CP[\boldsymbol{f}]$ as a system of equations with a nonnegativity condition on a variable vector $\boldsymbol{z} \in R^m$:

$$\boldsymbol{F}(\boldsymbol{z}) = \boldsymbol{0} \quad \text{and} \quad \boldsymbol{z} \geq \boldsymbol{0}, \tag{37}$$

and proposed a homotopy continuation method, $ALG1[U, \beta, \chi, \phi]$, founded on a one-parameter family of systems of equations:

$$\boldsymbol{F}(\boldsymbol{z}) = t\boldsymbol{c} \quad \text{and} \quad \boldsymbol{z} \geq \boldsymbol{0}.$$

Here $m = 2n$. Supposing that the $CP[\boldsymbol{f}]$ satisfies Condition 1.5 and (i) of Condition 5.1, we have seen that the system above enjoys the following properties:

(a) We can choose a $\boldsymbol{c} > \boldsymbol{0}$, a closed convex cone $U \subset R^m_{++} \cup \{\boldsymbol{0}\}$ and a point $\boldsymbol{z}^1 \in \boldsymbol{F}^{-1}(U)$ such that (ii), (iii) and (iv) of Condition 5.1 hold.
(b) $R^m_{++} \subset \boldsymbol{F}(R^m_{++})$.
(c) $\boldsymbol{F}$ maps R^m_{++} onto $\boldsymbol{F}(R^m_{++})$ diffeomorphically.
(d) the set $\boldsymbol{F}^{-1}(D)$ is bounded for every bounded subset D of R^m_{+}.

Generally, $ALG1[U, \beta, \chi, \phi]$ computes an approximate solution of the system (37) if all the conditions above are satisfied. This may remind the readers of applications of $ALG1[U, \beta, \chi, \phi]$ to some other problems which are converted into systems of the form (37).

The algorithms $ALG1[U, \beta, \chi, \phi]$ and $ALG2[U, \bar{\beta}, \chi, \phi]$ as well as their global and local convergence results in this paper can apply to linear complementarity problems satisfying Condition 1.5. We could modify the $ALG1[U, \beta, \chi, \phi]$ to derive the path-following algorithm ([13; 21]) which solves linear complementarity problems with positive semi-definite matrices in $O(\sqrt{n}L)$ iterations. Furthermore we could prove the globally linear convergence of the modified algorithm when it is applied to linear complementarity problems with P-matrices. These results on the positive semi-definite and P-matrix cases, which were presented in the original version of the paper [8] but cut in the revised version, will be further extended to a wider subclass of linear complementarity problems with P_0-matrices in the paper [9] where we explore a unified approach ([14]) to both the path-following algorithm ([13; 21]) and the potential reduction algorithm ([14]) for linear complementarity problems.

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