

INTEGER PROGRAMMING (IP)

SYSTEM OF LINEAR INEQUALITIES

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n &\geq b_1 \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n &\geq b_2 \end{aligned}$$

$$a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n \geq b_m$$

CS154EXTRA NOTES - WEEK 8

OR  $A \cdot \vec{x} \geq \vec{b}$

ALL  $a_{ij}$  /  $b_k$  ARE INTEGERS

IP INSTANCE INTEGER  $m \times n$  ARRAY A,  
INTEGER  $n$ -VECTOR b.

PROBLEM DOES THERE EXIST INTEGER  $n$ -VECTOR  
 $\vec{x}$  SUCH THAT  $A \cdot \vec{x} \geq \vec{b}$

THEM IP IS NP-COMPLETE

PROOF 1) IP ENP — USES NON-TRIVIAL LINEAR ALGEBRA (SEE TEXT)

2) IP IS NP-HARD — 3-SAT  $\leq_{poly}$  IP

REDN A INPUT  $F(x_1, \dots, x_n) = c_1 \wedge c_2 \wedge \dots \wedge c_m$   
WITH  $c_i = z_{i1} \vee z_{i2} \vee z_{i3}$

OUTPUT INSTANCE OF IP

INSTANCE?

VARIABLES  $x_1, \bar{x}_1, x_2, \bar{x}_2, \dots, x_n, \bar{x}_n$   
INEQUALITIES 2 TYPES

A FOR  $1 \leq i \leq n$

$$\begin{cases} \cdot x_i + \bar{x}_i \geq 1 \\ \cdot -x_i - \bar{x}_i \geq -1 \\ \cdot x_i \geq 0 \\ \cdot \bar{x}_i \geq 0 \end{cases}$$

# ONLY WAY TO SATISFY THESE INEQUALITIES  
→ MAKE ONE OF  $x_i / \bar{x}_i$  EQUAL 1

& THE OTHER EQUAL 0

D FOR  $c_j = z_{j1} \vee z_{j2} \vee z_{j3}$

$$\begin{cases} \cdot z_{j1} + z_{j2} + z_{j3} \geq 1 \end{cases}$$

# ONLY WAY TO SATISFY THIS IS TO MAKE  
AT LEAST ONE OF  $z_{ji}$ 'S EQUAL 1

ONE VIEW VALUE 1 AS "TRUE", 0 AS "FALSE"

F IS POLY-TIME COMPUTABLE

IF F IS SATISFIABLE  $\Leftrightarrow$

IP HAS SOLUTION.

(12)

PROOF ( $\Rightarrow$ )

GIVEN T.A. FOR F  
MAKE  $x_i$  (OR  $\bar{x}_i$ ) EQUAL 1 IF T.A.  
GIVES IT TRUE, ELSE MAKE IT 0

CLEARLY ALL TYPE A INEQUALITIES ARE  
MET.

ALSO SINCE T.A. MAKES ALL CLAUSES TRUE  
TYPE B CAUSES NONE ALSO MET.

DONE

PROOF ( $\Leftarrow$ )

SUPPOSE I.P. HAS SOLUTION  
BUT T.A. MUST BE 0-1 SOLUTION &  
 $x_i \neq \bar{x}_i$

REPLACE 1 BY TRUE, 0 BY FALSE, TO  
GET LEGAL T.A.

CLAIM T.A. SATISFIES F, BY CLAIM 2

DONE

THUS IP IS NP-COMPLETE.