CS156: The Calculus of Computation

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It is reasonable to hope that the relationship between **computation** *and* **mathematical logic** *will be as fruitful in the next century as that between* **analysis** *and* **physics** *in the last. The development of this relationship demands a concern for both applications and mathematical elegance.*

John McCarthy

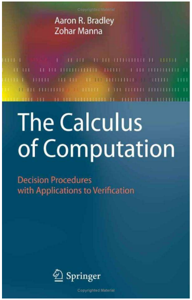
A Basis for a Mathematical Theory of Computation, 1963

Textbook

The Calculus of Computation:
Decision Procedures with
Applications to Verification

by
Aaron Bradley
Zohar Manna

Springer 2007



Topics: Overview

1. First-Order logic
2. Specification and verification
3. Satisfiability decision procedures

Part I: Foundations

1. Propositional Logic
2. First-Order Logic
3. First-Order Theories
4. Induction
5. Program Correctness: Mechanics
   Inductive assertion method, Ranking function method

Part II: Decision Procedures

7. Quantified Linear Arithmetic
   Quantifier elimination for integers and rationals
8. Quantifier-Free Linear Arithmetic
   Linear programming for rationals
9. Quantifier-Free Equality and Data Structures
10. Combining Decision Procedures
    Nelson-Oppen combination method
11. Arrays
    More than quantifier-free fragment

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Motivation

Motivation I

Decision Procedures are algorithms to decide formulae.
These formulae can arise

- in software verification.
- in hardware verification

Consider the following program:

for
  @ $\ell \leq i \leq u \land (rv \leftrightarrow \exists j.\ \ell \leq j < i \land a[j] = e)$
  (int $i := \ell$; $i \leq u$; $i := i + 1$) {
  if ($a[i] = e$) $rv :=$ true;
}

How can we decide whether the formula is a loop invariant?

Motivation II

Prove: (Path 1)

$$\texttt{assume}\ \ell \le i \le u \wedge (rv \leftrightarrow \exists j.\ \ell \le j < i \wedge a[j] = e)$$
$$\texttt{assume}\ i \le u$$
$$\texttt{assume}\ a[i] = e$$
$$rv := \texttt{true};$$
$$i := i + 1$$
$$@\ \ell \le i \le u \wedge (rv \leftrightarrow \exists j.\ \ell \le j < i \wedge a[j] = e)$$

Motivation III

Path 2:

$$\texttt{assume}\ \ell \le i \le u \wedge (rv \leftrightarrow \exists j.\ \ell \le j < i \wedge a[j] = e)$$
$$\texttt{assume}\ i \le u$$
$$\texttt{assume}\ a[i] \neq e$$
$$i := i + 1$$
$$@\ \ell \le i \le u \wedge (rv \leftrightarrow \exists j.\ \ell \le j < i \wedge a[j] = e)$$

Each path generates a Verification Condition (VC). We have to prove that each VC holds (valid).

Motivation IV

The VC for path 1 is computed by substitution:

$$\texttt{assume}\ \ell \le i \le u \wedge (rv \leftrightarrow \exists j.\ \ell \le j < i \wedge a[j] = e)$$
$$\texttt{assume}\ i \le u$$
$$\texttt{assume}\ a[i] = e$$
$$rv := \texttt{true};$$
$$i := i + 1$$
$$@\ \ell \le i \le u \wedge (rv \leftrightarrow \exists j.\ \ell \le j < i \wedge a[j] = e)$$

Substituting $\top$ for rv and $i+1$ for i, the postcondition (denoted by the @ symbol) holds if and only if the VC:

$$\ell \le i \le u \wedge (rv \leftrightarrow \exists j.\ \ell \le j < i \wedge a[j] = e) \wedge i \le u \wedge a[i] = e$$
$$\rightarrow \ell \le i + 1 \le u \wedge (\top \leftrightarrow \exists j.\ \ell \le j < i + 1 \wedge a[j] = e)$$

holds.

Motivation V

We need an algorithm that decides whether this formula holds. If the formula does not hold, the algorithm should give a counterexample; e.g.,

$$\ell = 0, i = 1, u = 1, rv = \mathit{false}, a[0] = 0, a[1] = 1, e = 1.$$

We will discuss such algorithms in later lectures.

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Chapter 1: Propositional Logic (PL)

Propositional Logic (PL)

PL Syntax

Atom	truth symbols $\top$ ("true") and $\bot$ ("false")
	propositional variables $P, Q, R, P_1, Q_1, R_1, \ldots$
Literal	atom α or its negation $\neg\alpha$
Formula	literal or application of a logical connective to formulae F, F_1, F_2

$\neg F$	"not"	(negation)
$F_1 \wedge F_2$	"and"	(conjunction)
$F_1 \vee F_2$	"or"	(disjunction)
$F_1 \rightarrow F_2$	"implies"	(implication)
$F_1 \leftrightarrow F_2$	"if and only if"	(iff)

Example:

formula $F : (P \wedge Q) \rightarrow (\top \vee \neg Q)$
atoms: P, Q, $\top$
literals: P, Q, $\top$, $\neg Q$
subformulae: P, Q, $\top$, $\neg Q$, $P \wedge Q$, $\top \vee \neg Q$, F
abbreviation

$$F : P \wedge Q \rightarrow \top \vee \neg Q$$

PL Semantics (meaning of PL)

Formula F + Interpretation I = Truth value (true, false)

Interpretation

$$I : \{P \mapsto \text{true}, Q \mapsto \text{false}, \cdots\}$$

Evaluation of F under I:

F	$\neg F$
0	1
1	0

where 0 corresponds to value false
1 true

F_1	F_2	$F_1 \wedge F_2$	$F_1 \vee F_2$	$F_1 \rightarrow F_2$	$F_1 \leftrightarrow F_2$
0	0	0	0	1	1
0	1	0	1	1	0
1	0	0	1	0	0
1	1	1	1	1	1

Example:

$F : P \wedge Q \;\rightarrow\; P \vee \neg Q$

$I : \{P \mapsto \text{true}, Q \mapsto \text{false}\}$ i.e., $I[P] = \text{true}, I[Q] = \text{false}$

P	Q	$\neg Q$	$P \wedge Q$	$P \vee \neg Q$	F
1	0	1	0	1	1

$1 = \text{true}$ $0 = \text{false}$

F evaluates to true under I; i.e., $I[F] = \text{true}$.

Inductive Definition of PL's Semantics

$I \models F$ if F evaluates to true under I

$I \not\models F$ false

Base Case:

$I \models \top$ $I \not\models \bot$

$I \models P$ iff $I[P] = \text{true}$; i.e., P is true under I

$I \not\models P$ iff $I[P] = \text{false}$

Inductive Case:

$I \models \neg F$ iff $I \not\models F$

$I \models F_1 \wedge F_2$ iff $I \models F_1$ and $I \models F_2$

$I \models F_1 \vee F_2$ iff $I \models F_1$ or $I \models F_2$ (or both)

$I \models F_1 \rightarrow F_2$ iff $I \models F_1$ implies $I \models F_2$

$I \models F_1 \leftrightarrow F_2$ iff, $I \models F_1$ and $I \models F_2$,
or $I \not\models F_1$ and $I \not\models F_2$

Note:

$I \models F_1 \rightarrow F_2$ iff $I \not\models F_1$ or $I \models F_2$.

$I \not\models F_1 \rightarrow F_2$ iff $I \models F_1$ and $I \not\models F_2$.

$I \not\models F_1 \vee F_2$ iff $I \not\models F_1$ and $I \not\models F_2$.

Example of Inductive Reasoning:

$F :\; P \wedge Q \;\rightarrow\; P \vee \neg Q$

$I :\; \{P \;\mapsto\; \text{true},\; Q \;\mapsto\; \text{false}\}$

1. $I \models P$ since $I[P] = \text{true}$
2. $I \not\models Q$ since $I[Q] = \text{false}$
3. $I \models \neg Q$ by 2 and $\neg$
4. $I \not\models P \wedge Q$ by 2 and $\wedge$
5. $I \models P \vee \neg Q$ by 1 and $\vee$
6. $I \models F$ by 4 and $\rightarrow$ Why?

Thus, F is true under I.

Note: steps 1, 3, and 5 are nonessential.

Satisfiability and Validity

F satisfiable iff there exists an interpretation I such that $I \models F$.

F valid iff for all interpretations I, $I \models F$.

F is valid iff $\neg F$ is unsatisfiable

Goal: devise an algorithm to decide validity or unsatisfiability of formula F.

Method 1: Truth Tables

Example $\quad F : P \wedge Q \;\rightarrow\; P \vee \neg Q$

P	Q	$P \wedge Q$	$\neg Q$	$P \vee \neg Q$	F
0	0	0	1	1	1
0	1	0	0	0	1
1	0	0	1	1	1
1	1	1	0	1	1

Thus F is valid.

Example $\quad F : P \vee Q \;\rightarrow\; P \wedge Q$

P	Q	$P \vee Q$	$P \wedge Q$	F	
0	0	0	0	1	← satisfying I
0	1	1	0	0	← falsifying I
1	0	1	0	0	
1	1	1	1	1	

Thus F is satisfiable, but invalid.

Method 2: Semantic Argument

- Assume F is not valid and I a falsifying interpretation: $I \not\models F$
- Apply proof rules.
- If no contradiction reached and no more rules applicable, F is invalid.
- If in every branch of proof a contradiction reached, F is valid.

Proof Rules for Semantic Arguments I

$$\frac{I \models \neg F}{I \not\models F} \qquad \frac{I \not\models \neg F}{I \models F}$$

$$\frac{I \models F \wedge G}{\begin{array}{l} I \models F \\ I \models G \end{array}} \;\leftarrow\text{and} \qquad \frac{I \not\models F \wedge G}{I \not\models F \;\mid\; I \not\models G} \;\nwarrow\text{or}$$

$$\frac{I \models F \vee G}{I \models F \;\mid\; I \models G} \qquad \frac{I \not\models F \vee G}{\begin{array}{l} I \not\models F \\ I \not\models G \end{array}}$$

Proof Rules for Semantic Arguments II

$$\frac{I \models F \rightarrow G}{I \not\models F \;\mid\; I \models G} \qquad \frac{I \not\models F \rightarrow G}{\begin{array}{l} I \models F \\ I \not\models G \end{array}}$$

$$\frac{I \models F \leftrightarrow G}{I \models F \wedge G \;\mid\; I \not\models F \vee G} \qquad \frac{I \not\models F \leftrightarrow G}{I \models F \wedge \neg G \;\mid\; I \models \neg F \wedge G}$$

$$\frac{\begin{array}{l} I \models F \\ I \not\models F \end{array}}{I \models \bot}$$

Example: Prove

$$F: \ P \wedge Q \ \rightarrow \ P \vee \neg Q \quad \text{is valid.}$$

Let's assume that F is not valid and that I is a falsifying interpretation.

1.	$I \not\models P \wedge Q \ \rightarrow \ P \vee \neg Q$	assumption
2.	$I \models P \wedge Q$	1 and $\rightarrow$
3.	$I \not\models P \vee \neg Q$	1 and $\rightarrow$
4.	$I \models P$	2 and $\wedge$
5.	$I \not\models P$	3 and $\vee$
6.	$I \models \bot$	4 and 5 are contradictory

Thus F is valid.

Example: Prove

$$F: \ (P \ \rightarrow \ Q) \wedge (Q \ \rightarrow \ R) \ \rightarrow \ (P \ \rightarrow \ R) \quad \text{is valid.}$$

Let's assume that F is not valid.

1.	$I \not\models F$	assumption
2.	$I \models (P \ \rightarrow \ Q) \wedge (Q \ \rightarrow \ R)$	1 and $\rightarrow$
3.	$I \not\models P \ \rightarrow \ R$	1 and $\rightarrow$
4.	$I \models P$	3 and $\rightarrow$
5.	$I \not\models R$	3 and $\rightarrow$
6.	$I \models P \ \rightarrow \ Q$	2 and $\wedge$
7.	$I \models Q \ \rightarrow \ R$	2 and $\wedge$

6.	$I \models P \ \rightarrow \ Q$	2 and $\wedge$
7.	$I \models Q \ \rightarrow \ R$	2 and $\wedge$
8a.	$I \not\models P$	6 and $\rightarrow$ (case a)
9a.	$I \models \bot$	4 and 8
8b.	$I \models Q$	6 and $\rightarrow$ (case b)
9ba.	$I \not\models Q$	7 and $\rightarrow$ (subcase ba)
10ba.	$I \models \bot$	8b and 9ba
9bb.	$I \models R$	7 and $\rightarrow$ (subcase bb)
10bb.	$I \models \bot$	5 and 9bb
9b.	$I \models \bot$	10ba and 10bb
8.	$I \models \bot$	9a and 9b

Our assumption is contradictory in all cases, so F is valid.

Example 3: Is

$$F: \ P \vee Q \ \rightarrow \ P \wedge Q$$

valid? Assume F is not valid:

1.	$I \not\models P \vee Q \ \rightarrow \ P \wedge Q$	assumption
2.	$I \models P \vee Q$	1 and $\rightarrow$
3.	$I \not\models P \wedge Q$	1 and $\rightarrow$
4a.	$I \models P$	2, $\vee$ (case a)
5aa.	$I \not\models P$	3, $\vee$ (subcase aa)
6aa.	$I \models \bot$	4a, 5aa
5ab.	$I \not\models Q$	3, $\vee$ (subcase ab)
6ab.	?	
5a.	?	

4b.	$I \models Q$	2, $\lor$ (case b)
5ba.	$I \not\models P$	3, $\lor$ (subcase ba)
6ba.	?	
5bb.	$I \not\models Q$	3, $\lor$ (subcase bb)
6bb.	$I \models \bot$	4b, 5bb
5b.	?	
5.	?	

We cannot derive a contradiction in both cases (4a and 4b), so we cannot prove that F is valid. To demonstrate that F is not valid, however, we must find a falsifying interpretation (here are two):

$I_1 : \{P \mapsto \text{true},\ Q \mapsto \text{false}\} \qquad I_2 : \{Q \mapsto \text{true},\ P \mapsto \text{false}\}$

Note: we have to derive a contradiction in all cases for F to be valid!

Equivalence

F_1 and F_2 are equivalent ($F_1 \Leftrightarrow F_2$)
iff for all interpretations I, $I \models F_1 \leftrightarrow F_2$

To prove $F_1 \Leftrightarrow F_2$, show $F_1 \leftrightarrow F_2$ is valid, that is, both $F_1 \rightarrow F_2$ and $F_2 \rightarrow F_1$ are valid.

F_1 entails F_2 ($F_1 \Rightarrow F_2$)
iff for all interpretations I, $I \models F_1 \rightarrow F_2$

Note: $F_1 \Leftrightarrow F_2$ and $F_1 \Rightarrow F_2$ are not formulae!!

Example: Show

$$P \rightarrow Q \Leftrightarrow \neg P \lor Q$$

i.e.

$$F : (P \rightarrow Q) \leftrightarrow (\neg P \lor Q) \textit{ is valid.}$$

Assume F is not valid, then we have two cases:

Case a: $I \not\models \neg P \lor Q$,
$I \models P \rightarrow Q$

Case b: $I \models \neg P \lor Q$,
$I \not\models P \rightarrow Q$

Derive contradictions in both cases.

Normal Forms

1. Negation Normal Form (NNF)
   $\neg, \land, \lor$ are the only boolean connectives allowed.
   Negations may occur only in literals of the form $\neg P$.
   To transform F into equivalent F' in NNF, apply the following template equivalences recursively (and left-to-right):

$$\neg\neg F_1 \Leftrightarrow F_1 \qquad \neg\top \Leftrightarrow \bot \qquad \neg\bot \Leftrightarrow \top$$

$$\left.\begin{array}{l}\neg(F_1 \land F_2) \Leftrightarrow \neg F_1 \lor \neg F_2 \\ \neg(F_1 \lor F_2) \Leftrightarrow \neg F_1 \land \neg F_2\end{array}\right\}\text{De Morgan's Law}$$

$$F_1 \rightarrow F_2 \Leftrightarrow \neg F_1 \lor F_2$$

$$F_1 \leftrightarrow F_2 \Leftrightarrow (F_1 \rightarrow F_2) \land (F_2 \rightarrow F_1)$$

"Complete" syntactic restriction: every F has an equivalent F' in NNF.

Example: Convert

$$F: \neg(P \rightarrow \neg(P \land Q))$$

to NNF.

$$F' : \neg(\neg P \lor \neg(P \land Q)) \qquad \rightarrow$$
$$F'' : \neg\neg P \land \neg\neg(P \land Q) \qquad \text{De Morgan's Law}$$
$$F''' : P \land P \land Q \qquad \neg\neg$$

F''' is equivalent to F ($F''' \Leftrightarrow F$) and is in NNF.

2. Disjunctive Normal Form (DNF)
Disjunction of conjunctions of literals

$$\bigvee_i \bigwedge_j \ell_{i,j} \quad \text{for literals } \ell_{i,j}$$

To convert F into equivalent F' in DNF,
transform F into NNF and then
use the following template equivalences (left-to-right):

$$\left.\begin{array}{lcl} (F_1 \lor F_2) \land F_3 & \Leftrightarrow & (F_1 \land F_3) \lor (F_2 \land F_3) \\ F_1 \land (F_2 \lor F_3) & \Leftrightarrow & (F_1 \land F_2) \lor (F_1 \land F_3) \end{array}\right\} \mathit{dist}$$

Note: formulae can grow exponentially as the distributivity laws are applied.

Example: Convert

$$F: (Q_1 \lor \neg\neg Q_2) \land (\neg R_1 \rightarrow R_2)$$

into equivalent DNF

$$F' : (Q_1 \lor Q_2) \land (R_1 \lor R_2) \qquad \text{in NNF}$$
$$F'' : (Q_1 \land (R_1 \lor R_2)) \lor (Q_2 \land (R_1 \lor R_2)) \qquad \text{dist}$$
$$F''' : (Q_1 \land R_1) \lor (Q_1 \land R_2) \lor (Q_2 \land R_1) \lor (Q_2 \land R_2) \qquad \text{dist}$$

F''' is equivalent to F ($F''' \Leftrightarrow F$) and is in DNF.

3. Conjunctive Normal Form (CNF)

Conjunction of disjunctions of literals

$$\bigwedge_i \bigvee_j \ell_{i,j} \quad \text{for literals } \ell_{i,j}$$

To convert F into equivalent F' in CNF,
transform F into NNF and then
use the following template equivalences (left-to-right):

$$\begin{array}{lcl} (F_1 \land F_2) \lor F_3 & \Leftrightarrow & (F_1 \lor F_3) \land (F_2 \lor F_3) \\ F_1 \lor (F_2 \land F_3) & \Leftrightarrow & (F_1 \lor F_2) \land (F_1 \lor F_3) \end{array}$$

A disjunction of literals is called a clause.

Example: Convert

$$F : P \leftrightarrow (Q \rightarrow R)$$

to an equivalent formula F' in CNF.

First get rid of $\leftrightarrow$:

$$F_1 : (P \rightarrow (Q \rightarrow R)) \land ((Q \rightarrow R) \rightarrow P)$$

Now replace $\rightarrow$ with $\lor$:

$$F_2 : (\neg P \lor (\neg Q \lor R)) \land (\neg(\neg Q \lor R) \lor P)$$

Drop unnecessary parentheses and apply De Morgan's Law:

$$F_3 : (\neg P \lor \neg Q \lor R) \land ((\neg\neg Q \land \neg R) \lor P)$$

Simplify double negation (now in NNF):

$$F_4 : (\neg P \lor \neg Q \lor R) \land ((Q \land \neg R) \lor P)$$

Distribute disjunction over conjunction (now in CNF):

$$F' : (\neg P \lor \neg Q \lor R) \land (Q \lor P) \land (\neg R \lor P)$$

Equisatisfiability

Definition

F and F' are *equisatisfiable*, iff

F is satisfiable if and only if F' is satisfiable

Every formula is equisatifiable to either $\top$ or $\bot$.

Goal: Decide satisfiability of PL formula F

Step 1: Convert F to equisatisfiable formula F' in CNF

Step 2: Decide satisfiability of formula F' in CNF

Step 1: Convert F to equisatisfiable formula F' in CNF I

There is an *efficient conversion* of F to F' where

- F' is in CNF and
- F and F' are equisatisfiable

Note: efficient means polynomial in the size of F.

Basic Idea:

- Introduce a new variable P_G for every subformula G of F, unless G is already an atom.

Step 1: Convert F to equisatisfiable formula F' in CNF II

- For each subformula

  $$G : G_1 \circ G_2,$$

  produce a small formula

  $$P_G \leftrightarrow P_{G_1} \circ P_{G_2}.$$

  Here $\circ$ denotes an arbitrary connective ($\neg$, $\lor$, $\land$, $\rightarrow$, $\leftrightarrow$); if the connective is $\neg$, G_1 should be ignored.

Step 1: Convert F to equisatisfiable formula F' in CNF III

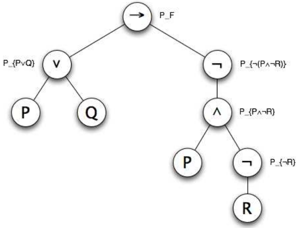

Figure: Parse tree for $F : P \vee Q \;\rightarrow\; \neg(P \wedge \neg R)$

Step 1: Convert F to equisatisfiable formula F' in CNF IV

- Convert each of these (small) formulae separately to an equivalent CNF formula

$$\mathrm{CNF}(P_G \;\leftrightarrow\; P_{G_1} \circ P_{G_2})\;.$$

Let S_F be the set of all non-atom subformulae G of F (including F itself). The formula

$$P_F \wedge \bigwedge_{G \in S_F} CNF(P_G \;\leftrightarrow\; P_{G_1} \circ P_{G_2})$$

is equisatisfiable to F. (Why?)

The number of subformulae is linear in the size of F.
The time to convert one small formula is constant!

Example: CNF I

Convert

$$F:\; P \vee Q \;\rightarrow\; P \wedge \neg R$$

to an equisatisfiable formula in CNF.

Introduce new variables: P_F, $P_{P\vee Q}$, $P_{P\wedge\neg R}$, $P_{\neg R}$.

Create new formulae and convert them to equivalent formulae in CNF separately:

- $F_1 = \mathrm{CNF}(P_F \;\leftrightarrow\; (P_{P\vee Q} \;\rightarrow\; P_{P\wedge\neg R}))$:

$$(\neg P_F \vee \neg P_{P\vee Q} \vee P_{P\wedge\neg R}) \wedge (P_F \vee P_{P\vee Q}) \wedge (P_F \vee \neg P_{P\wedge\neg R})$$

- $F_2 = \mathrm{CNF}(P_{P\vee Q} \;\leftrightarrow\; P \vee Q)$:

$$(\neg P_{P\vee Q} \vee P \vee Q) \wedge (P_{P\vee Q} \vee \neg P) \wedge (P_{P\vee Q} \vee \neg Q)$$

Example: CNF II

- $F_3 = \mathrm{CNF}(P_{P\wedge\neg R} \;\leftrightarrow\; P \wedge P_{\neg R})$:

$$(\neg P_{P\wedge\neg R} \vee P) \wedge (\neg P_{P\wedge\neg R} \vee P_{\neg R}) \wedge (P_{P\wedge\neg R} \vee \neg P \vee \neg P_{\neg R})$$

- $F_4 = \mathrm{CNF}(P_{\neg R} \;\leftrightarrow\; \neg R)$:

$$(\neg P_{\neg R} \vee \neg R) \wedge (P_{\neg R} \vee R)$$

$P_F \wedge F_1 \wedge F_2 \wedge F_3 \wedge F_4$ is in CNF and equisatisfiable to F.

Step 2: Decide the satisfiability of PL formula F' in CNF

Boolean Constraint Propagation (BCP)

If a clause contains one literal ℓ,

Set ℓ to $\top$:	$\cdots \wedge \overset{\top}{\ell} \wedge \cdots$
Remove all clauses containing ℓ:	$\cdots \wedge \cancel{(\cdots \vee \ell \vee \cdots)} \wedge \cdots$
Remove $\neg\ell$ in all clauses:	$\cdots \wedge (\cdots \vee \cancel{\neg\ell} \vee \cdots) \wedge \cdots$

based on the unit resolution

$$\frac{\ell \qquad \neg\ell \vee C}{C} \quad \leftarrow \text{clause}$$

Pure Literal Propagation (PLP)

If P occurs only positive (without negation), set it to $\top$.
If P occurs only negative set it to $\bot$.
Then do the simplifications as in Boolean Constraint Propagation

Davis-Putnam-Logemann-Loveland (DPLL) Algorithm

Decides the satisfiability of PL formulae in CNF

Decision Procedure DPLL: Given F in CNF

```
let rec DPLL F =
  let F' = BCP F in
  let F'' = PLP F' in
  if F'' = ⊤ then true
  else if F'' = ⊥ then false
  else
     let P = CHOOSE vars(F'') in
     (DPLL F''{P ↦ ⊤}) ∨ (DPLL F''{P ↦ ⊥})
```


Simplification

Simplify according to the template equivalences (left-to-right) [exercise 1.2]

$\neg\bot \Leftrightarrow \top$	$\neg\top \Leftrightarrow \bot$	$\neg\neg F \Leftrightarrow F$
$F \wedge \top \Leftrightarrow F$	$F \wedge \bot \Leftrightarrow \bot$	$\cdots$
$F \vee \top \Leftrightarrow \top$	$F \vee \bot \Leftrightarrow F$	$\cdots$

Example I

Consider

$$F:\ (\neg P \vee Q \vee R) \wedge (\neg Q \vee R) \wedge (\neg Q \vee \neg R) \wedge (P \vee \neg Q \vee \neg R).$$

Branching on Q

On the first branch, we have

$$F\{Q \mapsto \top\}:\ (R) \wedge (\neg R) \wedge (P \vee \neg R).$$

By unit resolution,

$$\frac{R \qquad (\neg R)}{\bot},$$

so $F\{Q \mapsto \top\} = \bot \Rightarrow$ false.

Example II

Recall

$$F: (\neg P \lor Q \lor R) \land (\neg Q \lor R) \land (\neg Q \lor \neg R) \land (P \lor \neg Q \lor \neg R).$$

On the other branch, we have

$$F\{Q \mapsto \bot\}: (\neg P \lor R).$$

Furthermore, by PLP,

$$F\{Q \mapsto \bot,\ R \mapsto \top\} = \top \Rightarrow \text{true}$$

Thus F is satisfiable with satisfying interpretation

$$I: \{P \mapsto \text{false},\ Q \mapsto \text{false},\ R \mapsto \text{true}\}.$$

or

$$I: \{P \mapsto \text{true},\ Q \mapsto \text{false},\ R \mapsto \text{true}\}.$$

Example

$$F: (\neg P \lor Q \lor R) \land (\neg Q \lor R) \land (\neg Q \lor \neg R) \land (P \lor \neg Q \lor \neg R)$$