CS156: The Calculus of Computation

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It is reasonable to hope that the relationship between **computation** *and* **mathematical logic** *will be as fruitful in the next century as that between* **analysis** *and* **physics** *in the last. The development of this relationship demands a concern for both applications and mathematical elegance.*

John McCarthy
A Basis for a Mathematical Theory of Computation, 1963

Textbook

THE CALCULUS OF COMPUTATION:
Decision Procedures with
Applications to Verification

by
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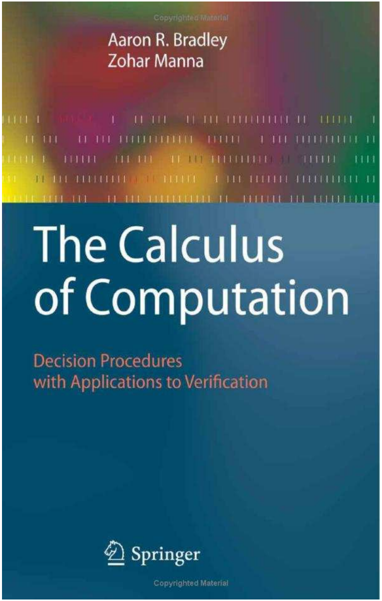
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Topics: Overview

1. First-Order logic
2. Specification and verification
3. Satisfiability decision procedures

Part I: Foundations

1. Propositional Logic
2. First-Order Logic
3. First-Order Theories
4. Induction
5. Program Correctness: Mechanics
   Inductive assertion method, Ranking function method

Part II: Decision Procedures

7. Quantified Linear Arithmetic
   Quantifier elimination for integers and rationals
8. Quantifier-Free Linear Arithmetic
   Linear programming for rationals
9. Quantifier-Free Equality and Data Structures
10. Combining Decision Procedures
    Nelson-Oppen combination method
11. Arrays
    More than quantifier-free fragment

Motivation

Motivation I

Decision Procedures are algorithms to decide formulae.
These formulae can arise

- in software verification.
- in hardware verification

Consider the following program:

---

for
  @ $\ell \leq i \leq u \land (rv \;\leftrightarrow\; \exists j.\ \ell \leq j < i \land a[j] = e)$
  $($`int` $i := \ell;\ i \leq u;\ i := i + 1) \{$
  `if` $(a[i] = e)\ rv :=$ `true`;
$\}$

How can we decide whether the formula is a loop invariant?

Motivation II

Prove: (Path 1)

$\texttt{assume}\ \ell \leq i \leq u \wedge (rv \ \leftrightarrow\ \exists j.\ \ell \leq j < i \wedge a[j] = e)$

$\texttt{assume}\ i \leq u$

$\texttt{assume}\ a[i] = e$

$rv := \texttt{true};$

$i := i + 1$

$\texttt{@}\ \ell \leq i \leq u \wedge (rv \ \leftrightarrow\ \exists j.\ \ell \leq j < i \wedge a[j] = e)$

Motivation III

Path 2:

$\texttt{assume}\ \ell \leq i \leq u \wedge (rv \ \leftrightarrow\ \exists j.\ \ell \leq j < i \wedge a[j] = e)$

$\texttt{assume}\ i \leq u$

$\texttt{assume}\ a[i] \neq e$

$i := i + 1$

$\texttt{@}\ \ell \leq i \leq u \wedge (rv \ \leftrightarrow\ \exists j.\ \ell \leq j < i \wedge a[j] = e)$

Each path generates a Verification Condition (VC). We have to prove that each VC holds (valid).

Motivation IV

The VC for path 1 is computed by substitution:

$$\texttt{assume}\ \ell \le i \le u \wedge (rv \ \leftrightarrow\ \exists j.\ \ell \le j < i \wedge a[j] = e)$$
$$\texttt{assume}\ i \le u$$
$$\texttt{assume}\ a[i] = e$$
$$rv := \texttt{true};$$
$$i := i + 1$$
$$\texttt{@}\ \ell \le i \le u \wedge (rv \ \leftrightarrow\ \exists j.\ \ell \le j < i \wedge a[j] = e)$$

Substituting $\top$ for rv and $i+1$ for i, the postcondition (denoted by the @ symbol) holds if and only if the VC:

$$\begin{aligned} &\ell \le i \le u \wedge (rv \ \leftrightarrow\ \exists j.\ \ell \le j < i \wedge a[j] = e) \wedge i \le u \wedge a[i] = e \\ \to\ &\ell \le i+1 \le u \wedge (\top \ \leftrightarrow\ \exists j.\ \ell \le j < i+1 \wedge a[j] = e) \end{aligned}$$

holds.

Motivation V

We need an algorithm that decides whether this formula holds. If the formula does not hold, the algorithm should give a counterexample; e.g.,

$$\ell = 0, i = 1, u = 1, rv = \mathit{false}, a[0] = 0, a[1] = 1, e = 1.$$

We will discuss such algorithms in later lectures.

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Chapter 1: Propositional Logic (PL)

Propositional Logic (PL)

PL Syntax

Atom	truth symbols $\top$ ("true") and $\bot$ ("false")
	propositional variables $P, Q, R, P_1, Q_1, R_1, \ldots$
Literal	atom α or its negation $\neg\alpha$
Formula	literal or application of a
	logical connective to formulae F, F_1, F_2

$\neg F$	"not"	(negation)
$F_1 \wedge F_2$	"and"	(conjunction)
$F_1 \vee F_2$	"or"	(disjunction)
$F_1 \rightarrow F_2$	"implies"	(implication)
$F_1 \leftrightarrow F_2$	"if and only if"	(iff)

Example:

formula $F : (P \wedge Q) \;\rightarrow\; (\top \vee \neg Q)$
atoms: $P,\ Q,\ \top$
literals: $P,\ Q,\ \top,\ \neg Q$
subformulae: $P,\ Q,\ \top,\ \neg Q,\ P \wedge Q,\ \top \vee \neg Q,\ F$
abbreviation

$$F : P \wedge Q \;\rightarrow\; \top \vee \neg Q$$

PL Semantics (meaning of PL)

Formula F + Interpretation I = Truth value (true, false)

Interpretation

$$I : \{P \mapsto \text{true}, Q \mapsto \text{false}, \cdots\}$$

Evaluation of F under I:

F	$\neg F$
0	1
1	0

where 0 corresponds to value false
1 true

F_1	F_2	$F_1 \wedge F_2$	$F_1 \vee F_2$	$F_1 \rightarrow F_2$	$F_1 \leftrightarrow F_2$
0	0	0	0	1	1
0	1	0	1	1	0
1	0	0	1	0	0
1	1	1	1	1	1

Example:

$F: P \wedge Q \;\rightarrow\; P \vee \neg Q$
$I: \{P \mapsto \text{true}, Q \mapsto \text{false}\}$ i.e., $I[P] = \text{true}, I[Q] = \text{false}$

P	Q	$\neg Q$	$P \wedge Q$	$P \vee \neg Q$	F
1	0	1	0	1	1

$1 = \text{true}$ $0 = \text{false}$

F evaluates to true under I; i.e., $I[F] = \text{true}$.

Inductive Definition of PL's Semantics

$$\begin{array}{llll} I \models F & \text{if } F \text{ evaluates to} & \text{true} & \text{under } I \\ I \not\models F & & \text{false} & \end{array}$$

<u>Base Case</u>:

$$I \models \top \qquad I \not\models \bot$$

$$\begin{array}{lll} I \models P & \text{iff} & I[P] = \text{true; i.e., } P \text{ is true under } I \\ I \not\models P & \text{iff} & I[P] = \text{false} \end{array}$$

<u>Inductive Case</u>:

$$\begin{array}{ll} I \models \neg F & \text{iff } I \not\models F \\ I \models F_1 \wedge F_2 & \text{iff } I \models F_1 \text{ and } I \models F_2 \\ I \models F_1 \vee F_2 & \text{iff } I \models F_1 \text{ or } I \models F_2 \text{ (or both)} \\ I \models F_1 \rightarrow F_2 & \text{iff } I \models F_1 \text{ implies } I \models F_2 \\ I \models F_1 \leftrightarrow F_2 & \text{iff, } I \models F_1 \text{ and } I \models F_2, \\ & \quad \text{or } I \not\models F_1 \text{ and } I \not\models F_2 \end{array}$$

<u>Note</u>:

$I \models F_1 \rightarrow F_2$ iff $I \not\models F_1$ or $I \models F_2$.

$I \not\models F_1 \rightarrow F_2$ iff $I \models F_1$ and $I \not\models F_2$.

$I \not\models F_1 \vee F_2$ iff $I \not\models F_1$ and $I \not\models F_2$.

Example of Inductive Reasoning:

$$F : \; P \wedge Q \;\rightarrow\; P \vee \neg Q$$

$$I : \; \{P \;\mapsto\; \text{true}, \;\; Q \;\mapsto\; \text{false}\}$$

1.	I	$\models$	P	since $I[P] = \text{true}$	
2.	I	$\not\models$	Q	since $I[Q] = \text{false}$	
3.	I	$\models$	$\neg Q$	by 2 and $\neg$	
4.	I	$\not\models$	$P \wedge Q$	by 2 and $\wedge$	
5.	I	$\models$	$P \vee \neg Q$	by 1 and $\vee$	
6.	I	$\models$	F	by 4 and $\rightarrow$	Why?

Thus, F is true under I.

Note: steps 1, 3, and 5 are nonessential.

Satisfiability and Validity

F satisfiable iff there exists an interpretation I such that $I \models F$.
F valid iff for all interpretations I, $I \models F$.

$$\boxed{F \text{ is valid iff } \neg F \text{ is unsatisfiable}}$$

Goal: devise an algorithm to decide validity or unsatisfiability of formula F.

Method 1: Truth Tables

Example $F : P \wedge Q \;\rightarrow\; P \vee \neg Q$

P	Q	$P \wedge Q$	$\neg Q$	$P \vee \neg Q$	F
0	0	0	1	1	1
0	1	0	0	0	1
1	0	0	1	1	1
1	1	1	0	1	1

Thus F is valid.

Example $F : P \vee Q \;\rightarrow\; P \wedge Q$

P	Q	$P \vee Q$	$P \wedge Q$	F	
0	0	0	0	1	$\leftarrow$ satisfying I
0	1	1	0	0	$\leftarrow$ falsifying I
1	0	1	0	0	
1	1	1	1	1	

Thus F is satisfiable, but invalid.

Method 2: Semantic Argument

- Assume F is not valid and I a falsifying interpretation:
  $I \not\models F$
- Apply proof rules.
- If no contradiction reached and no more rules applicable,
  F is invalid.
- If in every branch of proof a contradiction reached,
  F is valid.

Proof Rules for Semantic Arguments I

$$\frac{I \models \neg F}{I \not\models F} \qquad \frac{I \not\models \neg F}{I \models F}$$

$$\frac{I \models F \wedge G}{\begin{array}{l} I \models F \\ I \models G \end{array}} \longleftarrow \text{and} \qquad \frac{I \not\models F \wedge G}{I \not\models F \quad \mid \quad I \not\models G}$$

$$\nwarrow \text{or}$$

$$\frac{I \models F \vee G}{I \models F \quad \mid \quad I \models G} \qquad \frac{I \not\models F \vee G}{\begin{array}{l} I \not\models F \\ I \not\models G \end{array}}$$

Proof Rules for Semantic Arguments II

$$\frac{I \models F \rightarrow G}{I \not\models F \quad | \quad I \models G}$$

$$\frac{I \not\models F \rightarrow G}{\begin{array}{l} I \models F \\ I \not\models G \end{array}}$$

$$\frac{I \models F \leftrightarrow G}{I \models F \wedge G \quad | \quad I \not\models F \vee G}$$

$$\frac{I \not\models F \leftrightarrow G}{I \models F \wedge \neg G \quad | \quad I \models \neg F \wedge G}$$

$$\frac{\begin{array}{l} I \models F \\ I \not\models F \end{array}}{I \models \bot}$$

Example: Prove

$$F : \; P \land Q \;\rightarrow\; P \lor \neg Q \quad \text{is valid.}$$

Let's assume that F is not valid and that I is a falsifying interpretation.

1.	I	$\not\models$	$P \land Q \;\rightarrow\; P \lor \neg Q$	assumption
2.	I	$\models$	$P \land Q$	1 and $\rightarrow$
3.	I	$\not\models$	$P \lor \neg Q$	1 and $\rightarrow$
4.	I	$\models$	P	2 and $\land$
5.	I	$\not\models$	P	3 and $\lor$
6.	I	$\models$	$\perp$	4 and 5 are contradictory

Thus F is valid.

Example: Prove

$$F : (P \rightarrow Q) \wedge (Q \rightarrow R) \rightarrow (P \rightarrow R) \quad \text{is valid.}$$

Let's assume that F is not valid.

1.	$I \not\models F$	assumption
2.	$I \models (P \rightarrow Q) \wedge (Q \rightarrow R)$	1 and $\rightarrow$
3.	$I \not\models P \rightarrow R$	1 and $\rightarrow$
4.	$I \models P$	3 and $\rightarrow$
5.	$I \not\models R$	3 and $\rightarrow$
6.	$I \models P \rightarrow Q$	2 and $\wedge$
7.	$I \models Q \rightarrow R$	2 and $\wedge$

6.	$I \models P \rightarrow Q$	2 and $\wedge$
7.	$I \models Q \rightarrow R$	2 and $\wedge$
8a.	$I \not\models P$	6 and $\rightarrow$ (case a)
9a.	$I \models \bot$	4 and 8
8b.	$I \models Q$	6 and $\rightarrow$ (case b)
9ba.	$I \not\models Q$	7 and $\rightarrow$ (subcase ba)
10ba.	$I \models \bot$	8b and 9ba
9bb.	$I \models R$	7 and $\rightarrow$ (subcase bb)
10bb.	$I \models \bot$	5 and 9bb
9b.	$I \models \bot$	10ba and 10bb
8.	$I \models \bot$	9a and 9b

Our assumption is contradictory in all cases, so F is valid.

Example 3: Is

$$F: \; P \lor Q \;\rightarrow\; P \land Q$$

valid? Assume F is not valid:

1.	$I \not\models P \lor Q \;\rightarrow\; P \land Q$	assumption
2.	$I \models P \lor Q$	1 and $\rightarrow$
3.	$I \not\models P \land Q$	1 and $\rightarrow$
4a.	$I \models P$	2, $\lor$ (case a)
5aa.	$I \not\models P$	3, $\lor$ (subcase aa)
6aa.	$I \models \bot$	4a, 5aa
5ab.	$I \not\models Q$	3, $\lor$ (subcase ab)
6ab.	?	
5a.	?	

4b.	$I \models Q$	2, $\vee$ (case b)
5ba.	$I \not\models P$	3, $\vee$ (subcase ba)
6ba.	?	
5bb.	$I \not\models Q$	3, $\vee$ (subcase bb)
6bb.	$I \models \perp$	4b, 5bb
5b.	?	
5.	?	

We cannot derive a contradiction in both cases (4a and 4b), so we cannot prove that F is valid. To demonstrate that F is not valid, however, we must find a falsifying interpretation (here are two):

$I_1 : \{P \mapsto \text{true}, \ Q \mapsto \text{false}\} \qquad I_2 : \{Q \mapsto \text{true}, \ P \mapsto \text{false}\}$

Note: we have to derive a contradiction in all cases for F to be valid!

Equivalence

F_1 and F_2 are <u>equivalent</u> ($F_1 \Leftrightarrow F_2$)

iff for all interpretations I, $I \models F_1 \leftrightarrow F_2$

To prove $F_1 \Leftrightarrow F_2$, show $F_1 \leftrightarrow F_2$ is valid, that is, both $F_1 \rightarrow F_2$ and $F_2 \rightarrow F_1$ are valid.

F_1 <u>entails</u> F_2 ($F_1 \Rightarrow F_2$)

iff for all interpretations I, $I \models F_1 \rightarrow F_2$

<u>Note</u>: $F_1 \Leftrightarrow F_2$ and $F_1 \Rightarrow F_2$ are not formulae!!

Example: Show

$$P \rightarrow Q \Leftrightarrow \neg P \vee Q$$

i.e.

$$F : (P \rightarrow Q) \leftrightarrow (\neg P \vee Q) \textit{ is valid}.$$

Assume F is not valid, then we have two cases:

Case a: $I \not\models \neg P \vee Q$,

$I \models P \rightarrow Q$

Case b: $I \models \neg P \vee Q$,

$I \not\models P \rightarrow Q$

Derive contradictions in both cases.

Normal Forms

1. Negation Normal Form (NNF)

   $\neg, \wedge, \vee$ are the only boolean connectives allowed.

   Negations may occur only in literals of the form $\neg P$.

   To transform F into equivalent F' in NNF, apply the following template equivalences recursively (and left-to-right):

$$\neg\neg F_1 \;\Leftrightarrow\; F_1 \qquad \neg\top \;\Leftrightarrow\; \bot \qquad \neg\bot \;\Leftrightarrow\; \top$$

$$\left.\begin{array}{l} \neg(F_1 \wedge F_2) \;\Leftrightarrow\; \neg F_1 \vee \neg F_2 \\ \neg(F_1 \vee F_2) \;\Leftrightarrow\; \neg F_1 \wedge \neg F_2 \end{array}\right\}\text{De Morgan's Law}$$

$$F_1 \;\rightarrow\; F_2 \;\Leftrightarrow\; \neg F_1 \vee F_2$$

$$F_1 \;\leftrightarrow\; F_2 \;\Leftrightarrow\; (F_1 \;\rightarrow\; F_2) \wedge (F_2 \;\rightarrow\; F_1)$$

   "Complete" syntactic restriction: every F has an equivalent F' in NNF.

Example: Convert

$$F: \neg(P \rightarrow \neg(P \wedge Q))$$

to NNF.

$$F' : \neg(\neg P \vee \neg(P \wedge Q)) \qquad \rightarrow$$

$$F'' : \neg\neg P \wedge \neg\neg(P \wedge Q) \qquad \text{De Morgan's Law}$$

$$F''' : P \wedge P \wedge Q \qquad \neg\neg$$

F''' is equivalent to F ($F''' \Leftrightarrow F$) and is in NNF.

2. Disjunctive Normal Form (DNF)

Disjunction of conjunctions of literals

$$\bigvee_i \bigwedge_j \ell_{i,j} \quad \text{for literals } \ell_{i,j}$$

To convert F into equivalent F' in DNF,
transform F into NNF and then
use the following template equivalences (left-to-right):

$$\left.\begin{array}{lcl} (F_1 \vee F_2) \wedge F_3 & \Leftrightarrow & (F_1 \wedge F_3) \vee (F_2 \wedge F_3) \\ F_1 \wedge (F_2 \vee F_3) & \Leftrightarrow & (F_1 \wedge F_2) \vee (F_1 \wedge F_3) \end{array}\right\} \textit{dist}$$

Note: formulae can grow exponentially as the distributivity laws are applied.

Example: Convert

$$F: \ (Q_1 \vee \neg\neg Q_2) \wedge (\neg R_1 \ \rightarrow \ R_2)$$

into equivalent DNF

$$
\begin{array}{lll}
F' : (Q_1 \vee Q_2) \wedge (R_1 \vee R_2) & & \text{in NNF} \\
F'' : (Q_1 \wedge (R_1 \vee R_2)) \vee (Q_2 \wedge (R_1 \vee R_2)) & & \text{dist} \\
F''' : (Q_1 \wedge R_1) \vee (Q_1 \wedge R_2) \vee (Q_2 \wedge R_1) \vee (Q_2 \wedge R_2) & & \text{dist}
\end{array}
$$

F''' is equivalent to F ($F''' \Leftrightarrow F$) and is in DNF.

3. Conjunctive Normal Form (CNF)

Conjunction of disjunctions of literals

$$\bigwedge_i \bigvee_j \ell_{i,j} \quad \text{for literals } \ell_{i,j}$$

To convert F into equivalent F' in CNF,
transform F into NNF and then
use the following template equivalences (left-to-right):

$$\begin{array}{lcl}
(F_1 \wedge F_2) \vee F_3 & \Leftrightarrow & (F_1 \vee F_3) \wedge (F_2 \vee F_3) \\
F_1 \vee (F_2 \wedge F_3) & \Leftrightarrow & (F_1 \vee F_2) \wedge (F_1 \vee F_3)
\end{array}$$

A disjunction of literals is called a clause.

Example: Convert

$$F : P \leftrightarrow (Q \rightarrow R)$$

to an equivalent formula F' in CNF.

First get rid of $\leftrightarrow$:

$$F_1 : (P \rightarrow (Q \rightarrow R)) \wedge ((Q \rightarrow R) \rightarrow P)$$

Now replace $\rightarrow$ with $\vee$:

$$F_2 : (\neg P \vee (\neg Q \vee R)) \wedge (\neg(\neg Q \vee R) \vee P)$$

Drop unnecessary parentheses and apply De Morgan's Law:

$$F_3 : (\neg P \vee \neg Q \vee R) \wedge ((\neg\neg Q \wedge \neg R) \vee P)$$

Simplify double negation (now in NNF):

$$F_4 : (\neg P \vee \neg Q \vee R) \wedge ((Q \wedge \neg R) \vee P)$$

Distribute disjunction over conjunction (now in CNF):

$$F' : (\neg P \vee \neg Q \vee R) \wedge (Q \vee P) \wedge (\neg R \vee P)$$

Equisatisfiability

Definition

F and F' are *equisatisfiable*, iff

$$F \text{ is satisfiable if and only if } F' \text{ is satisfiable}$$

Every formula is equisatifiable to either $\top$ or $\bot$.

<u>Goal</u>: Decide satisfiability of PL formula F

<u>Step 1</u>: Convert F to equisatisfiable formula F' in CNF

<u>Step 2</u>: Decide satisfiability of formula F' in CNF

Step 1: Convert F to equisatisfiable formula F' in CNF I

There is an *efficient conversion* of F to F' where

- F' is in CNF and
- F and F' are equisatisfiable

Note: efficient means polynomial in the size of F.

Basic Idea:

- Introduce a new variable P_G for every subformula G of F, unless G is already an atom.

Step 1: Convert F to equisatisfiable formula F' in CNF II

- For each subformula

$$G : \; G_1 \circ G_2,$$

produce a small formula

$$P_G \; \leftrightarrow \; P_{G_1} \circ P_{G_2}.$$

Here $\circ$ denotes an arbitrary connective ($\neg$, $\vee$, $\wedge$, $\rightarrow$, $\leftrightarrow$); if the connective is $\neg$, G_1 should be ignored.

Step 1: Convert F to equisatisfiable formula F' in CNF III

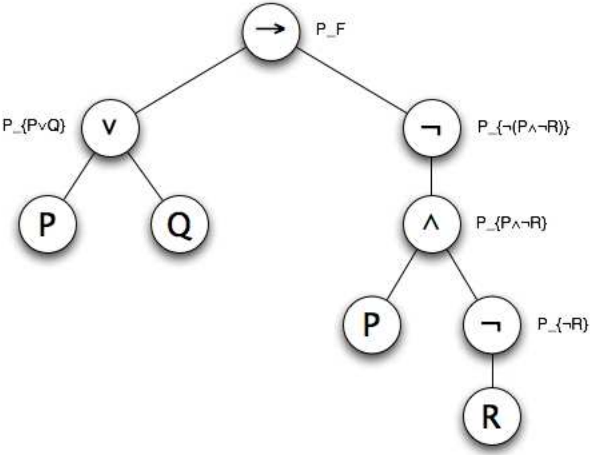

Figure: Parse tree for $F : P \vee Q \ \rightarrow \ \neg(P \wedge \neg R)$

Step 1: Convert F to equisatisfiable formula F' in CNF IV

- Convert each of these (small) formulae separately to an equivalent CNF formula

$$\mathrm{CNF}(P_G \leftrightarrow P_{G_1} \circ P_{G_2}) \ .$$

Let S_F be the set of all non-atom subformulae G of F (including F itself). The formula

$$P_F \wedge \bigwedge_{G \in S_F} CNF(P_G \leftrightarrow P_{G_1} \circ P_{G_2})$$

is equisatisfiable to F. (Why?)

The number of subformulae is linear in the size of F.
The time to convert one small formula is constant!

Example: CNF I

Convert

$$F: \; P \vee Q \; \rightarrow \; P \wedge \neg R$$

to an equisatisfiable formula in CNF.

Introduce new variables: P_F, $P_{P \vee Q}$, $P_{P \wedge \neg R}$, $P_{\neg R}$.

Create new formulae and convert them to equivalent formulae in CNF separately:

- $F_1 = \text{CNF}(P_F \; \leftrightarrow \; (P_{P \vee Q} \; \rightarrow \; P_{P \wedge \neg R}))$:

$$(\neg P_F \vee \neg P_{P \vee Q} \vee P_{P \wedge \neg R}) \wedge (P_F \vee P_{P \vee Q}) \wedge (P_F \vee \neg P_{P \wedge \neg R})$$

- $F_2 = \text{CNF}(P_{P \vee Q} \; \leftrightarrow \; P \vee Q)$:

$$(\neg P_{P \vee Q} \vee P \vee Q) \wedge (P_{P \vee Q} \vee \neg P) \wedge (P_{P \vee Q} \vee \neg Q)$$

Example: CNF II

- $F_3 = \text{CNF}(P_{P \wedge \neg R} \;\leftrightarrow\; P \wedge P_{\neg R})$:

$$(\neg P_{P \wedge \neg R} \vee P) \wedge (\neg P_{P \wedge \neg R} \vee P_{\neg R}) \wedge (P_{P \wedge \neg R} \vee \neg P \vee \neg P_{\neg R})$$

- $F_4 = \text{CNF}(P_{\neg R} \;\leftrightarrow\; \neg R)$:

$$(\neg P_{\neg R} \vee \neg R) \wedge (P_{\neg R} \vee R)$$

$P_F \wedge F_1 \wedge F_2 \wedge F_3 \wedge F_4$ is in CNF and equisatisfiable to F.

Step 2: Decide the satisfiability of PL formula F' in CNF

Boolean Constraint Propagation (BCP)

If a clause contains one literal ℓ,

Set ℓ to $\top$: $\cdots \wedge \overset{\top}{\cancel{\ell}} \wedge \cdots$

Remove all clauses containing ℓ: $\cdots \wedge \cancel{(\cdots \vee \ell \vee \cdots)} \wedge \cdots$

Remove $\neg\ell$ in all clauses: $\cdots \wedge (\cdots \vee \cancel{\neg\ell} \vee \cdots) \wedge \cdots$

based on the unit resolution

$$\frac{\ell \qquad \neg\ell \vee C}{C} \quad \leftarrow \text{clause}$$

Pure Literal Propagation (PLP)

If P occurs only positive (without negation), set it to $\top$.

If P occurs only negative set it to $\bot$.

Then do the simplifications as in Boolean Constraint Propagation

Davis-Putnam-Logemann-Loveland (DPLL) Algorithm

Decides the satisfiability of PL formulae in CNF

Decision Procedure DPLL: Given F in CNF

$$
\begin{array}{l}
\texttt{let rec } \mathrm{DPLL}\ F = \\
\quad \texttt{let } F' = \mathrm{BCP}\ F \texttt{ in} \\
\quad \texttt{let } F'' = \mathrm{PLP}\ F' \texttt{ in} \\
\quad \texttt{if } F'' = \top \texttt{ then true} \\
\quad \texttt{else if } F'' = \bot \texttt{ then false} \\
\quad \texttt{else} \\
\qquad \texttt{let } P = \mathrm{CHOOSE}\ \mathsf{vars}(F'') \texttt{ in} \\
\qquad (\mathrm{DPLL}\ F''\{P \mapsto \top\}) \vee (\mathrm{DPLL}\ F''\{P \mapsto \bot\})
\end{array}
$$

Simplification

Simplify according to the template equivalences (left-to-right) [exercise 1.2]

$$
\begin{array}{rcl@{\qquad\qquad}rcl@{\qquad\qquad}rcl}
\neg\bot & \Leftrightarrow & \top & \neg\top & \Leftrightarrow & \bot & \neg\neg F & \Leftrightarrow & F \\
F \wedge \top & \Leftrightarrow & F & F \wedge \bot & \Leftrightarrow & \bot & & \cdots & \\
F \vee \top & \Leftrightarrow & \top & F \vee \bot & \Leftrightarrow & F & & \cdots &
\end{array}
$$

Example I

Consider

$$F: \ (\neg P \vee Q \vee R) \wedge (\neg Q \vee R) \wedge (\neg Q \vee \neg R) \wedge (P \vee \neg Q \vee \neg R).$$

<u>Branching on Q</u>

On the first branch, we have

$$F\{Q \ \mapsto \ \top\}: \ (R) \wedge (\neg R) \wedge (P \vee \neg R).$$

By unit resolution,

$$\frac{R \qquad (\neg R)}{\bot},$$

so $F\{Q \ \mapsto \ \top\} \ = \ \bot \ \Rightarrow$ false.

Example II

Recall

$$F: \ (\neg P \vee Q \vee R) \wedge (\neg Q \vee R) \wedge (\neg Q \vee \neg R) \wedge (P \vee \neg Q \vee \neg R).$$

On the other branch, we have

$$F\{Q \ \mapsto \ \bot\}: \ (\neg P \vee R).$$

Furthermore, by PLP,

$$F\{Q \ \mapsto \ \bot, \ R \ \mapsto \ \top\} \ = \ \top \ \Rightarrow \ \text{true}$$

Thus F is satisfiable with satisfying interpretation

$$I: \ \{P \ \mapsto \ \text{false}, \ Q \ \mapsto \ \text{false}, \ R \ \mapsto \ \text{true}\}.$$

or

$$I: \ \{P \ \mapsto \ \text{true}, \ Q \ \mapsto \ \text{false}, \ R \ \mapsto \ \text{true}\}.$$

Example

$$F:\ (\neg P \lor Q \lor R) \land (\neg Q \lor R) \land (\neg Q \lor \neg R) \land (P \lor \neg Q \lor \neg R)$$

$$\begin{tikzcd}
 & F \arrow[dl, dash, "Q \mapsto \top"'] \arrow[dr, dash, "Q \mapsto \bot"] & \\
(R) \land (\neg R) \land (P \lor \neg R) \arrow[d, dash] & & (\neg P \lor R) \arrow[d, dash, "R \mapsto \top"] \\
\bot & & \top
\end{tikzcd}$$

$$I:\ \{P \mapsto \text{false},\ Q \mapsto \text{false},\ R \mapsto \text{true}\}$$

(No matter what P is)